Trigonometric identity

Here is a short proof of the sin-product identity without almost any computation. It is due to Mario Bonk from U. Michigan.

The polynomial $P(x) = x^{n-1} + x^{n-2} + ... + 1$ has the n-th roots of unity different from 1 as zeros. Put $x = y + 1$, then $P(x) = P(y + 1) = Q(y)$, and Q is a monic polynomial with absolute coefficient n. This coefficient is the product of the zeros of Q (up to a sign), which are $x_k - 1$ where x_k runs through the roots of unity different from 1, i.e. $x_k = \exp(2\pi ik/n), k = 1, ..., n-1$. Taking absolut values in this product identity we obtain

$$n = \prod_{k=1}^{n-1} |\exp(2\pi ik/n) - 1| = 2^{n-1} \prod_{k=1}^{n-1} \sin(\pi k/n).$$

QED