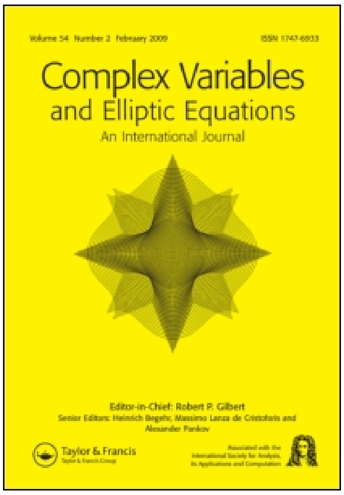

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An Analogue of the Defect Relation for the Uniform Metric

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Let f be a meromorphic function in the plane, denote by $A(r, f)$ the spherical area of $f(\{z : |z| \le r\})$ divided by the area of the Riemann sphere. For $a \in \boldsymbol{C}$ put

$$M(r, f) = M(r, \infty, f) = \sup_{\theta} |f(re^{i\theta})|, \quad M(r, a, f) = M(r, (f - a)^{-1})$$

and

$$b(a, f) = \liminf_{r \to \infty} \frac{\log^+ M(r, a, f)}{A(r, f)}.$$

Put $\boldsymbol{B}(f) = \{a : b(a, f) > 0\}$. Then the set $\boldsymbol{B}(f)$ is at most countable for every meromorphic function f. If there exists a_0 such that $b(a_0, f) > 2\pi$ then $\boldsymbol{B}(f) = \{a_0\}$. Otherwise

$$\sum_{a \in \bar{\boldsymbol{C}}} b(a, f) \le 2\pi. \tag{1}$$

For functions f of order $\lambda > 1/2$ we always have (1) and more, for any set $\{a_1, \dots, a_q\} \subset \bar{\boldsymbol{C}}$ there is a sequence $r_k \to \infty$ such that

$$\sum_{j=1}^{q} \log^+ M(r_k, a_j, f) \le (2\pi + o(1)) A(r_k, f).$$

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1. INTRODUCTION

We use the standard notations of the Nevanlinna Theory such as $T(r, f), N(r, f), N(r, a, f), m(r, f)$ and $m(r, a, f)$ (see [12]). In addition we use $M(r, f)$ and $M(r, a, f)$ defined in the Abstract.

The Second Main Theorem of Nevanlinna implies

$$\sum_{j=1}^{q} m(r, a_j, f) \leq (2 + o(1))T(r, f),$$

when $r \to \infty$ outside some exceptional set. This may be regarded as a bound for simultaneous approximation of several numbers $a_1, \ldots, a_q$ by f on the circumferences $\{z : |z| = r\}$ in L_1 metric. We are going to prove a similar statement for the uniform metric.

The history of the question starts from the work of R. Paley who made a conjecture that for entire functions f of order $\lambda > 1/2$

$$\liminf_{r\to\infty} \frac{\log^+ M(r, f)}{T(r, f)} \leq \pi\lambda. \tag{2}$$

In the case of $\lambda \leq 1/2$ the following inequality was proved for entire functions by G. Valiron in 1935 and for meromorphic functions by A. A. Goldberg and I. V. Ostrovskii in 1961:

$$\liminf_{r\to\infty} \frac{\log^+ M(r, f)}{T(r, f)} \leq \frac{\pi\lambda}{\sin \pi\lambda}. \tag{3}$$

The Paley conjecture was proved first by N. V. Govorov [4] and then V. P. Petrenko [13] discovered that (2) remains true for meromorphic functions. This result made natural the following definition

$$\beta(a, f) = \liminf_{r\to\infty} \frac{\log^+ M(r, a, f)}{T(r, f)},$$

which is analogous to Nevanlinna deficiency $\delta(a, f)$. We briefly summarize the main results on $\beta(a, f)$ obtained by V. P. Petrenko [14] and others. See also [4] which contains some development of V. P. Petrenko's work.

From the results mentioned above follows that $\beta(a, f) \leq \pi\lambda$, if $\lambda > 1/2$ and $\beta(a, f) \leq \pi\lambda \csc \pi\lambda$ if $\lambda \leq 1/2$. For meromorphic functions of finite (lower) order λ the set $\{a \in \bar{C} : \beta(a, f) > 0\}$ is at most countable and we have

$$\sum_{a \in \bar{C}} \beta(a, f) \leq \begin{cases} 2\pi\lambda, & \lambda > 1/2 \\ 2\pi\lambda \csc \pi\lambda, & \lambda \leq 1/2 \end{cases}$$

(This precise inequality is proved by I. I. Marchenko and A. I. Shcherba [10]). We also have

$$\sum_{a \in \bar{C}} \beta^{1/2}(a, f) < \infty,$$

proved in [7] and this relation is the best possible in a very strong sense: given a sequence $\{a_j\} \subset \bar{C}$ and a sequence of positive numbers β_j such that $\sum \beta_j^{1/2} < \infty$ there exists a meromorphic function f of (some) finite order such that $\beta(a_j, f) = \beta_j$ and $\beta(a, f) = 0$ if $a \notin \{a_j\}$. These and other results show that the behavior of $\beta(a, f)$ and $\delta(a, f)$ are similar for meromorphic functions of finite (lower) order.

For functions of infinite order the situation is quite different. There are examples of meromorphic functions f such that the set $\{a \in \bar{C} : \beta(a, f) = +\infty\}$ has the power of the continuum [14] (however the logarithmic capacity of $\{a \in \bar{C} : \beta(a, f) > 0\}$ is always zero).

Recently W. Bergweiler and H. Bock [2,3] found the way to formulate and prove a statement, analogous to the Paley conjecture, which remains valid for meromorphic functions of arbitrary fast growth: if the order of f is at least $1/2$ then

$$\liminf_{r \to \infty} \frac{\log^+ M(r, f)}{A(r, f)} \leq \pi. \tag{4}$$

We remark that $A(r, f) = dT(r, f)/d\log r + O(1)$ so for example if $T(r, f) \sim r^\lambda$ then $A(r, f) \sim \lambda r^\lambda$, which makes (4) plausible once (2) is known. Actually W. Bergweiler and H. Bock do not discuss the case of finite order in their paper, but their proof extends easily to the case of any order greater then $1/2$. Thus it is natural to define

$$b(a, f) = \liminf_{r \to \infty} \frac{\log^+ M(r, a, f)}{A(r, f)}.$$

Then (4) implies

$$b(a, f) \leq \pi$$

for every meromorphic function f of order greater then $1/2$ and every $a \in \bar{C}$. We have the following

THEOREM 1 *For every meromorphic function f of order at least $1/2$ and distinct points $a_1, \dots, a_q$ we have*

$$\sum_{j=1}^{q} \log^+ M(r, a_j, f) \leq (2\pi + o(1))A(r, f) \tag{5}$$

on some sequence of r tending to infinity.

Now we consider the case of meromorphic functions of order less than $1/2$. Here we can expect troubles because the precise estimate (3), on being divided by λ tends to ∞ when $\lambda \to 0$. Indeed, it may happen that $b(a, f) = \infty$ for functions of zero order. But we still can prove that the exceptional set $\boldsymbol{B}(f)$ is always countable.

Let us call $a \in \bar{C}$ a *strong exceptional value* if $b(a, f) > 0$ and in addition there exists a sequence $r_k \to \infty$ such that

$$f(r_k e^{i\theta}) \to a, \quad k \to \infty \tag{6}$$

uniformly with respect to $\theta \in [0, 2\pi]$. It is evident that if a function f has a strong exceptional value a then $\boldsymbol{B}(f) = \{a\}$, so card $B(f) = 1$ in this case.

THEOREM 2 *If f is a meromorphic function of order $\lambda < 1/2$ and $b(a, f) > \pi \sin \pi\lambda$ then a is a strong exceptional value.*

This theorem can be compared with the result of *O*. Teichmüller–A. A. Goldberg ([5, Ch. V, Theorem 3.1], and [6, Theorem 4.15]), which says that if $\delta(a, f) > 1 - \cos \pi\lambda$ and $\lambda < 1/2$ then (6) holds.

THEOREM 3 *If f is a meromorphic function of order $\lambda \leq 1/2$ without strong exceptional values, then*

$$\sum_{a \in \bar{C}} b(a, f) \leq 2\pi \sin \pi\lambda.$$

In particular (5) holds.

COROLLARY 7 *For every meromorphic function f the set $\boldsymbol{B}(f) = \{a \in \bar{C} : b(a, f) > 0\}$ is at most countable and either consists of one point a with $b(a, f) > 2\pi$ or*

$$\sum_{a \in \bar{C}} b(a, f) \leq 2\pi. \tag{7}$$

This result is the best possible. An evident example is $f(z) = e^z$. Furthermore, for every $\lambda \geq 1$ such that 2λ is an integer there are examples of meromorphic functions of order λ for which equality takes place in (7). These are the meromorphic functions which have precisely 2λ deficient values, each deficiency being equal to $1/\lambda$ (see [12, 5]). All such functions are precisely described in [9]. We conjecture that there are no other meromorphic functions of finite (lower) order for which the equality holds in (7). In particular for entire functions equality in (7) is likely only in the case of order 1.

Before starting the proof of the Theorem we recall some necessary facts from the theory of delta-subharmonic functions and from Ahlfors' "Uberlagerungsflachentheorie".

The author thanks Min Ru who asked the questions which stimulated this work and also W. Bergweiler for the discussion of the subject.

2. DELTA-SUBHARMONIC FUNCTIONS

A delta-subharmonic function is the difference of two subharmonic functions. We consider only delta-subharmonic functions in the plane $\boldsymbol{C}$ with the property $u(0) = 0$. By $\mu = \mu_u$ we denote the Riesz charge of u. We have the Jordan decomposition $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are Borel locally finite measures in the plane. We use the following notations:

$$\begin{aligned}
D(r) &= \{z : |z| \leq r\},\\
B_u(r) &= \sup_\theta u(re^{i\theta}),\\
m_u(r) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u^+(re^{i\theta})d\theta,\\
N_u(r) &= \int_0^r \mu^-(D(t))\frac{dt}{t},\\
T_u(r) &= m_u(r) + N_u(r).
\end{aligned}$$

The Poisson-Jensen Formula in these notations takes the form $T_u(r) = T_{-u}(r)$.

For every $r > 0$ denote by $R_u(re^{i\theta})$ the even function of θ, $|\theta| \leq \pi$, decreasing for $\theta \in [0, \pi]$ and equimeasurable with $\theta \mapsto u(re^{i\theta})$. This means that the sets $\{\theta \in [-\pi, \pi] : R_u(re^{i\theta}) > x\}$ and $\{\theta \in [-\pi, \pi] :$

$u(re^{i\theta}) > x\}$ have the same measure for all $x \in \boldsymbol{R}$. Define the Baernstein *-function [1] by

$$T_u^*(re^{i\phi}) = \frac{1}{\pi}\int_0^\phi R_u(re^{i\theta})d\theta + N_u(r).$$

This function is subharmonic in the upper half-plane, continuous in the closed upper half-plane and has the following properties:

$$T_u^*(r) = N_u(r), \quad r > 0, \tag{8}$$

$$\frac{\partial}{\partial\phi}T_u^*(r) = \frac{1}{\pi}B_u(r), \quad r > 0, \tag{9}$$

$$\frac{\partial}{\partial\phi}T_u^*(-r) = \frac{1}{\pi}\inf_\theta u(re^{i\theta}), \quad r > 0, \tag{10}$$

$$\max_\phi T_u^*(re^{i\phi}) = T_u(r). \tag{11}$$

In addition to these properties we will use the fact that the function $r \mapsto T^*(re^{i\theta})$ is convex with respect to the logarithm for every θ.

The following inequalities were used by M. Sodin [15] in his simplified proof of (2) and its generalization. Define

$$G_u(r, \lambda, \beta, \gamma) = \frac{1}{\pi}B_u(r)\cos\lambda\gamma + \lambda N_u(r)\sin\lambda\gamma -$$
$$- \frac{1}{\pi}R_u(re^{i\beta})\cos\lambda(\beta+\gamma) - \lambda T_u^*(re^{i\beta})\sin\lambda(\beta+\gamma). \tag{12}$$

LEMMA 1 *Let u be a delta-subharmonic function*, $0 < P < Q < Q' < \infty, 0 < \lambda < \infty, 0 < \beta < \pi$ *and* $|\beta + \gamma| \le \pi/(2\lambda)$. *Then*

$$\int_P^Q G_u(r, \lambda, \beta, \gamma)\frac{dr}{r^{\lambda+1}} \le \beta\lambda\left\{\frac{T_u(Q)}{Q^\lambda} + \frac{T_u(Q')}{\lambda Q^\lambda(\log Q' - \log Q)}\right\}, \tag{13}$$

Sketch of the proof. We assume that T^* is smooth in the closed upper half-plane. The general case is obtained using the standard approximation arguments given in [15]. Put $D = \{z : P < |z| < Q, 0 < \arg z < \beta\}$ and define $V(re^{i\theta}) = r^{-\lambda}\cos\lambda(\theta+\gamma)$. Apply the Second Green identity

$$\int_{\partial D}\left(T_u^*\frac{\partial V}{\partial n} - V\frac{\partial T_u^*}{\partial n}\right)ds = \int\int_D \left(T_u^*\Delta V - V\Delta T_u^*\right)d\sigma,$$

where $\partial/\partial n$ is differentiation in the direction of outer normal, ds is the length element and $d\sigma$ is the area element. As V is positive and harmonic in D and T^* is subharmonic, the right side is non-positive. Using the properties (8) and (9) of T^* we get

$$\int_P^Q G_u(r, \lambda, \beta, \gamma)\frac{dr}{r^{\lambda+1}} \leq \int_0^\beta \left\{Q^{-\lambda}\left(\lambda T_u^*(Qe^{i\theta}) + Q\frac{\partial}{\partial r}T_u^*(Qe^{i\theta})\right)\right.$$
$$\left. - P^{-\lambda}\left(\lambda T_u^*(Pe^{i\theta}) + P\frac{\partial}{\partial r}T_u^*(Pe^{i\theta})\right)\right\} \cos\lambda(\theta+\gamma)d\theta.$$

Now we perform the following transformations in the right side of this formula:

1. Drop the negative terms which contain P (we use the fact that $r \mapsto T_u^*(re^{i\theta})$ ia convex with respect to logarithm);
2. Drop $\cos\lambda(\theta+\gamma)$.
3. Use the estimate

$$Q\frac{\partial}{\partial r}T_u^*(Qe^{i\theta}) \leq \frac{1}{\log Q' - \log Q}\int_Q^{Q'} r\frac{\partial}{\partial r}T_u^*(re^{i\theta})d\log r$$
$$= \frac{1}{\log Q' - \log Q}T_u^*(Q'e^{i\theta}) \leq \frac{1}{\log Q' - \log Q}T_u(Q').$$

4. Replace T_u^* by T_u which is possible in view of (11).

Thus we obtain (13).

2. AHLFORS' THEORY AND POTENTIAL THEORY

Assume without loss of generality that the numbers $a_1, \ldots, a_q$ are finite. Choose δ, $0 < 3\delta < \min\{|a_i - a_j| : i \neq j\}$ and fix an arbitrarily large integer K.

Consider the delta-subharmonic functions

$$u_j(z) = \log^+ \frac{\delta}{|f(z) - a_j|}.$$

They have disjoint supports. A component D of the set $\{z : u_j(z) > 0\}$ is called *small* if:

(i) D contains at most K a_j-points of f, counting multiplicity, and
(ii) D is bounded or D is unbounded and $u_j(z) \to 0$ as $z \to \infty$, $z \in D$.

Otherwise the component D is called *large*. Note that the small components D are exactly those in which u_j is equal to the sum of at most K Green functions for D (with poles at a_j-points of f in D).

We write

$$u_j(z) = v_j(z) + w_j(z),$$

where v_j and w_j are non-negative delta-subharmonic functions with disjoint supports and $v_j(z) = u_j(z)$ if z belongs to a large component for u_j and $w_j(z) = u_j(z)$ if z belongs to a small component.

The following lemma belongs to H. Selberg (see for example [16, IV, 3]).

LEMMA 2

$$m_{w_j}(r) = O(1), \quad r \to \infty.$$

Now we put $\mu_j = \mu_{v_j}^+$ and estimate these measures from above. Our purpose is to show that

$$\sum_{j=1}^{q} \mu_j(D(r)) \leq 2A(r, f) + \text{ small error term }.$$

The following result is an easy exercise.

LEMMA 3 *Let Γ be an analytic curve dividing a disk D into two parts, D_1 and D_2. Let u_i be two functions harmonic in $\bar{D}_i$, $i = 1, 2$ and $u_1(z) = u_2(z)$, $z \in \Gamma$. Define*

$$u(z) = \begin{cases} u_1(z), & z \in D_1 \cup \Gamma \\ u_2(z), & z \in D_2 \end{cases}$$

Then u is delta-subharmonic with Riesz charge supported by Γ. Furthermore μ has a density $d(z)$ with respect to arc-length measure on Γ and

$$d(z) = \frac{1}{2\pi}\left(\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n}\right)(z),$$

where n is the unit normal to Γ, pointing from D_2 to D_1.

We conclude that μ_j is supported by some piecewise analytic curves where $|f(z) - a_j| = \delta$ and has density equal to

$$\frac{1}{2\pi}\left|\frac{\partial}{\partial n} \log \frac{\delta}{|f(z) - a_j|}\right|.$$

By the Cauchy-Riemann equations this is equal to

$$\frac{1}{2\pi}\left|\frac{\partial}{\partial s}\arg\frac{1}{f(z)-a_j}\right|,$$

where $\partial/\partial s$ is differentiation along the unit tangent vector to the level curve $|f(z)-a_j| = \delta$. Thus by integration along the boundaries with respect to $D(r)$ of all large components we get that $\mu_j(D(r))$ is equal to the *covering number* of the circle $\{\zeta : |\zeta - a_j| = \delta\}$ by boundaries of the large components of u_j in $D(r)$.

Here we are using the terminology and the main results of the Ahlfors Theory, which we briefly recall now (see [12] for proofs). We consider a holomorphic map of bordered Riemann surfaces $f : S \to S_0$. We assume that a smooth Riemannian metric ρ_0 is given on S_0 so that the area of S_0 is finite. We denote by ρ the pullback of ρ_0 via f. All lengths and areas will be measured with respect to these metrics ρ and ρ_0. If D is a Jordan region in S_0 and Γ is a rectifiable curve in S_0, the covering numbers over $A(D)$ and $A(\Gamma)$ are defined as

$$A(D) = \frac{\mathrm{area}(f^{-1}(D))}{\mathrm{area}(D)}$$

and

$$A(\Gamma) = \frac{\mathrm{length}(f^{-1}(\Gamma))}{\mathrm{length}(\Gamma)}.$$

Now define the length of the "relative boundary" as

$$L = \mathrm{length}(\partial S \cap f^{-1}(\mathrm{int}S_0)).$$

The First Main Theorem of Ahlfors says that for every smooth Jordan region $D \subset S_0$ there exists a constant h, depending only on D, S_0 and ρ_0, such that

$$|A(D) - A(S_0)| \leq hL.$$

Similarly for any smooth curve $\Gamma \subset S_0$ there exists a constant h depending only on Γ, S_0 and ρ_0, such that

$$|A(\Gamma) - A(S_0)| \leq hL.$$

We apply this to the case when S_0 is the closed disk $\Delta_j := \{\zeta : |\zeta - a_j| \leq \delta\}$ with the Euclidean metric, f is the meromorphic function which we are considering, and S is the intersection of the union of large components of u_j with the disk $D(r)$. Then all covering numbers will have parameter r and index j. We conclude that

$$\mu_j(D(r)) = A_j(r, \partial\Delta) \leq A_j(r, \Delta) + hL_j(r). \tag{14}$$

Now we apply Ahlfors' theory to the case when S_0 is the Riemann sphere with some smooth metric of finite area, coinciding with the Euclidean one on the disks Δ_j and $S = D(r)$. We denote the covering number of S_0 in this situation by $A(r)$ and the length of the relative boundary by $L(r)$. A component D of preimage $f^{-1}(\Delta_j)$ is called an *island* of multiplicity k if its closure is contained in $\text{int}(D(r))$ and f assumes a_j k times in D, counting multiplicity. A component D of preimage $f^{-1}(\Delta_j)$ is called a *peninsula* if it intersects $\partial D(r)$. It is clear that the intersection of a large component with $D(r)$ consists of islands of multiplicity at least $K + 1$ and of peninsulas.

The Second Main Theorem of Ahlfors implies that the sum of covering numbers of Δ_j by peninsulas and islands of multiplicity at least $K + 1$ does not exceed $2(1 + 1/K)A(r) + hL(r)$, where the constant h depends only on a_j, δ and the choice of the metric on the sphere (see [12]).We conclude from this and (14) that

$$\sum_{j=1}^{q} \mu_j(D(r)) \leq 2(1 + \frac{1}{K})A(r) + hL(r). \tag{15}$$

Now we notice that

$$T_{v_j}(r) = N_{-v_j}(r) = \int_0^r \mu_j(D(t))\frac{dt}{t}$$

because $v_j \geq 0$. So we integrate (15) and get

$$\sum_{j=1}^{q} T_{v_j}(r) \leq 2\left(1 + \frac{1}{K}\right) T(r) + S(r), \tag{16}$$

where $T(r) = T(r, f)$ and

$$S(r) = \int_0^r L(t)\frac{dt}{t}.$$

We have the following estimate for $S(r)$ [11]:

$$S(r) \leq h \log^{1/2} A(r)(T(r))^{1/2}. \tag{17}$$

3. BERGWEILER'S GENERALIZATION OF POLYA PEAKS

According to [3] there exist sequences $\lambda_k > 0$, $\rho_k \to \infty, M_k \to \infty$ and $\in_k \to 0$ such that

$$T(r) \leq (1+\in_k)T(\rho_k)\left(\frac{r}{\rho_k}\right)^{\lambda_k} \tag{18}$$

for

$$\left|\log \frac{r}{\rho_k}\right| \leq \frac{M_k}{\lambda_k}.$$

If the order λ of the function f is finite, we may take $\lambda_k = \lambda$, if $\lambda \neq 0$. Then (18) becomes the usual definition of Polya peaks. The case of zero order needs special consideration (see the Remark at the end of the paper). Furthermore, we may take

$$\lambda_k = o\left(\log^{3/2} T(\rho_k)\right). \tag{19}$$

Define t_k and T_k by

$$\log \frac{\rho_k}{t_k} = \log \frac{T_k}{\rho_k} = \frac{M_k}{\lambda_k}$$

so that (18) holds for $t_k \leq r \leq T_k$. Consider the set

$$A_k = \left\{ r : \rho_k \leq r \leq T_k, T(r) \leq \frac{1}{\sqrt{M_k}} T(\rho_k)\left(\frac{r}{\rho_k}\right)^{\lambda_k} \right\}$$

and define $Q'_k = T_k$ if $A_k = \emptyset$ and $Q'_k = \min A_k$ otherwise. Similarly we consider

$$B_k = \left\{ r : t_k \leq r \leq \rho_k, T(r) \leq \frac{1}{\sqrt{M_k}} T(\rho_k)\left(\frac{r}{\rho_k}\right)^{\lambda_k} \right\}$$

and define $P_k = t_k$ if $B_k = \emptyset$ and $P_k = \max B_k$ otherwise. We also define $Q_k = e^{-1/\lambda_k} Q'_k$. Then $t_k \leq P_k < \rho_k < Q_k < Q'_k \leq T_k$. The following estimates belong to W. Bergweiler and H. Bock [3]

LEMMA 4 *Denote*

$$I_k = \lambda_k \int_{P_k}^{Q_k} r^{-\lambda_k - 1} T(r) dr.$$

Then

$$T(P_k)P_k^{-\lambda_k} + T(Q_k')Q_k'^{-\lambda_k} = o(I_k), \quad k \to \infty.$$

We are going to apply Sodin's inequality (13) to the functions v_j and w_j with $P = P_k, Q = Q_k, Q' = Q_k'$ and $\lambda = \lambda_k$. We will estimate from above the error term (right side) of this inequality, using the evident relations

$$T_{w_j}(r) + T_{v_j}(r) = T_u(r) = T(r) + O(1).$$

For the first term in the right side of (13) we have

$$\frac{T(Q_k)}{Q_k^{\lambda_k}} \le e\frac{T(Q_k')}{Q_k'^{\lambda_k}} \tag{20}$$

by monotonicity of $T(r)$ and definition of Q_k. For the second term in the right side we have

$$\frac{T(Q_k')}{\lambda_k Q_k^{\lambda_k}(\log Q_k' - \log Q_k)} \le e\frac{T(Q_k')}{Q_k'^{\lambda_k}} \tag{21}$$

again by definition of Q_k.

Thus Lemmas 1 and 4 imply

$$\int_{P_k}^{Q_k} G_u(r, \lambda_k, \beta_k, \gamma_k)\frac{dr}{r^{\lambda_k+1}} + o(\beta_k\lambda_k I_k) \le 0. \tag{22}$$

Proof of Theorem 1. Choose in (22) $u = v_j, \gamma_k = 0$ and $\beta_k = \pi/(2\lambda_k)$ and replace $T^*_{v_j}$ by T_{v_j} in the expression (12). We obtain

$$\int_{P_k}^{Q_k} \left\{\frac{1}{\pi}B_{v_j}(r) - \lambda_k T_{v_j}(r)\right\}\frac{dr}{r^{\lambda_k+1}} \le o(I_k). \tag{23}$$

Now fix an arbitrarily small $\epsilon > 0$ and choose in (22) $u = w_j, \beta_k = \epsilon/\lambda_k$ and $\gamma_k = (\pi/2 - \epsilon)/\lambda_k$. Then divide the inequality by $\sin\epsilon$ and replace $T^*_{w_j}$ by T_{w_j} in the expression (12). We obtain

$$\int_{P_k}^{Q_k} \left\{\frac{1}{\pi}B_{w_j}(r) - \lambda_k\left(T_{w_j}(r)\csc\epsilon - N_{w_j}(r)\cot\epsilon\right)\right\}\frac{dr}{r^{\lambda_k+1}} \le o(I_k). \tag{24}$$

Now we use Lemma 2 to get $N_{w_j}(r) \ge T_{w_j}(r) - h$, where h is a constant. This permits us to replace in (24) $T_{w_j}\csc\epsilon - N_{w_j}\cot\epsilon$ by $T_{w_j}(1 -$

$\cos \in)\csc \in$. We remark that

$$B(r) := \sum_{j=1}^{q} \log^+ M(r, a_j, f) \leq \sum_{j=1}^{q} B_{u_j}(r) \leq \sum_{j=1}^{q} (B_{v_j}(r) + B_{w_j}(r)).$$

We add the inequalities (23) and (24) for $1 \leq j \leq q$ and use (16). We obtain

$$\int_{P_k}^{Q_k} \left\{ \frac{B(r)}{\pi} - \left(2\left(1 + \frac{1}{K}\right) + \frac{1 - \cos \in}{\sin \in} q \right) \lambda_k T(r) \right\} \frac{dr}{r^{\lambda_k + 1}}$$
$$\leq \int_{P_k}^{Q_k} \lambda_k S(r) \frac{dr}{r^{\lambda_k+1}} + o(I_k). \tag{25}$$

To estimate the integral in the right side we use (17) and the Minkowski inequality $ab \leq \frac{1}{3}a^3 + \frac{2}{3}b^{3/2}$:

$$S(r) \leq h\left(\log^{3/2} A(r) + T(r)^{3/4}\right)$$

Now using (19) and $ab \leq \dfrac{1}{2}(a^2 + b^2)$ we get

$$\lambda_k S(r) \leq h(\log^3 A(r) + \log^3 T(r) + o(\lambda_k T(r)))$$
$$= o(A(r) + \lambda_k T(r)). \tag{26}$$

Thus from (25) and (26) we obtain

$$\int_{P_k}^{Q_k} \left\{ \frac{B(r)}{\pi} - C(K, \in)\lambda_k T(r) + o(A(r)) \right\} \frac{dr}{r^{\lambda_k+1}} \leq 0,$$

where

$$C(K, \in) = 2\left(1 + \frac{1}{K}\right) + \frac{1 - \cos \in}{\sin \in} q + \in \to 2$$

as $\in \to 0$ and $K \to \infty$. Now we integrate by parts and use Lemma 4 and (20) to get

$$\int_{P_k}^{Q_k} \left\{ \frac{B(r)}{\pi} - C(K, \in)\lambda_k T(r) + o(A(r)) \right\} \frac{dr}{r^{\lambda_k+1}}$$
$$= C(K, \in)\left(\frac{T(P_k)}{P_k^{\lambda_k}} - \frac{T(Q_k)}{Q_k^{\lambda_k}} \right) + \int_{P_k}^{Q_k} \left\{ \frac{B(r)}{\pi} \right.$$
$$\left. -(C(K, \in) + o(1))A(r) \right\} \frac{dr}{r^{\lambda_k+1}}$$

So

$$\int_{P_k}^{Q_k} \left\{ \frac{B(r)}{\pi} - (C(K, \in) + o(1))A(r) \right\} \frac{dr}{r^{\lambda_k+1}} \leq 0.$$

Thus there exists a sequence $r_k \to \infty$ such that

$$B(r_k) \leq (C(K, \in) + o(1))A(r_k).$$

As K may be chosen arbitrarily large and $\in$ arbitrarily small, $C(K, \in)$ is arbitrarily close to 2 and this proves the theorem.

Proof of Theorem 2. Assume that for some j we have $b(a_j, f) > 0$ but a_j is not a *strict* exceptional value. We apply (22) with $u = v_j$, $\lambda_k = \lambda$, $\gamma_k = 0$ and $\beta_k = \pi$. But now we take into account that

$$\inf_\theta v_j(re^{i\theta}) = O(1), \quad r \to \infty.$$

This follows from the definition of a strict exceptional value. Thus we have in view of (10)

$$T^*_{v_j}(-r) = O(1), \quad r \to \infty.$$

So the term $R_{v_j}(re^{i\beta}) = R_{v_j}(-r)$ in the expression (12) for G_{v_j} disappears and we obtain

$$\int_{P_k}^{Q_k} \left\{ \frac{1}{\pi} B_{v_j}(r) - \lambda \sin \pi\lambda T_{v_j} \right\} \frac{dr}{r^{\lambda+1}} \leq o(I_k), \tag{27}$$

similarly to (23). Then we obtain (24), add it to (27) and integrate by parts like in the proof of Theorem 1. The result is

$$\int_{P_k}^{Q_k} \left\{ \frac{1}{\pi} B_{u_j}(r) - (\pi \sin \pi\lambda + \in)A(r) \right\} \frac{dr}{r^{\lambda+1}} \leq 0 \tag{28}$$

and we conclude that $b(a, f) \leq \pi \sin \pi\lambda$, which proves the theorem.

Proof of Theorem 3. Add the inequalities (27) and (24) for $1 \leq j \leq q$ and use (15). The error term $S(r)$ in (15) is $o(T(r))$ for functions of finite order [11]. Thus we obtain

$$\int_{P_k}^{Q_k} \left\{ \frac{1}{\pi} B(r) - C_1(K, \in)\lambda T(r) \right\} \frac{dr}{r^{\lambda+1}} \leq 0,$$

where

$$C_1(K, \in) = 2\pi \sin \pi\lambda \left(1 + \frac{1}{K}\right) + \frac{1 - \cos \in}{\sin \in} q + \in \to 2\pi \sin \pi\lambda < 2\pi$$

as $K \to \infty$ and $\epsilon \to 0$. Then we integrate by parts and finish the proof in the same way as for Theorem 1.

Remarks on the case of zero order. When the order of f is zero we don't need Polya peaks. We take an arbitrarily small number $\lambda > 0$, fix large P and instead of integrals from P_k to Q_k consider integrals from P to ∞. All integrals are convergent because $T(r) = o(r^\lambda)$ and the error terms tend to zero.

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