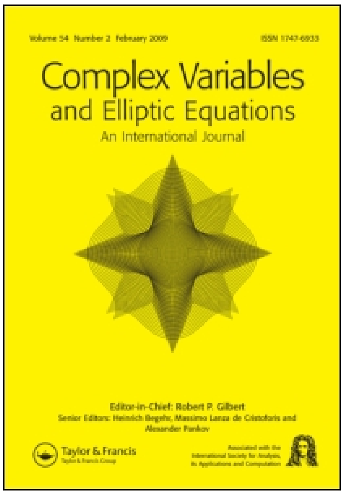

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An Analogue of the Defect Relation for the Uniform Metric, II*

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Let f be a meromorphic function in the plane, denote by $A(r, f)$ the spherical area of $f(\{z : |z| \le r\})$ divided by the area of the Riemann sphere. For $a \in \boldsymbol{C}$ put

$$M(r, f) = M(r, \infty, f) = \sup_{\theta} |f(re^{i\theta})|, \quad M(r, a, f) = M(r, (f - a)^{-1})$$

and

$$b(a, f) = \liminf_{r\to\infty} \frac{\log^+ M(r, a, f)}{A(r, f)}.$$

Put $\boldsymbol{B}(f) = \{a : b(a, f) > 0\}$. Assume that card $\boldsymbol{B}(f) > 1$. The recent results of W. Bergweiler, H. Bock and the author show that in this case $b(a, f) \le \pi, a \in \bar{\boldsymbol{C}}$, the set $\boldsymbol{B}(f)$ is at most countable, and

$$\sum_{a\in\bar{C}} b(a, f) \le 2\pi. \tag{1}$$

We show that for functions f of finite lower order the equality in (1) implies

$$\lim_{r\to\infty} \frac{\log T(r, f)}{\log r} = \frac{n}{2},$$

where $n \ge 2$ is an integer, and $b(a, f) = 2\pi/n$ for every $a \in \boldsymbol{B}(f)$.

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1. INTRODUCTION

We use the standard notations of the Nevanlinna Theory such as $T(r, f)$, $N(r, f)$, $N(r, a, f)$, $m(r, f)$, $m(r, a, f)$ and $\delta(a, f)$ (see [14]). In addition we use $M(r, f)$, $M(r, a, f)$ and $b(a, f)$ defined in the Abstract. Assume that the exceptional set $\boldsymbol{B}(f)$ contains at least two points. Then it is known [3] that

$$b(a, f) \leq \pi, \quad a \in \bar{\boldsymbol{C}}$$

and we have the following analogue [5] of Nevanlinna's Defect Relation

$$\sum_{a \in \bar{C}} b(a, f) \leq 2\pi. \tag{2}$$

In this paper we study the case of equality in (2).

THEOREM 1 *Let f be a meromorphic function of finite lower order, $\boldsymbol{B}(f)$ contains at least two points and equality takes place in (2). Then the limit exists*

$$\lim_{r \to \infty} \frac{\log T(r, f)}{\log r} = \frac{n}{2}, \tag{3}$$

where $n \geq 2$ is an integer, and $b(a, f) = 2\pi/n$, $a \in \boldsymbol{B}(f)$.

COROLLARY 1 *If f is a meromorphic function of finite lower order with $b(0, f) = b(\infty, f) = \pi$ then $\log T(t, f) \sim \log r$, $r \to \infty$.*

Theorem 1 is analogous to the following theorem of D. Drasin, confirming a conjecture of F. Nevanlinna of 1929.

THEOREM 2 *Let f be a meromorphic function of finite lower order with the property*

$$\sum_{a \in \bar{C}} \delta(a, f) = 2. \tag{4}$$

Then

$$T(r, f) = l(r) r^{n/2}, \tag{5}$$

where l is a slowly varying function in the sense of Karamata, $n \geq 2$ is an integer and $\delta(a, f) = 2p(a)/n$ with some integers $p(a)$.

(The regularity property (5) is not stated explicitly in Drasin's paper [4]. A simplified proof of his theorem, including (5) was given in [6]. Later in [7] the conclusions of Theorem 2 were derived from a weaker assumption then (4)).

Instead of $b(a, f)$ one can consider V. P. Petrenko's "deviations"

$$\beta(a, f) = \liminf_{r \to \infty} \frac{\log^+ M(r, a, f)}{T(r, f)}.$$

Recall that for meromorphic functions of lower order $1/2 < \lambda < \infty$ we have $\beta(a, f) \leq \pi\lambda$, $a \in \bar{C}$ [15] and

$$\sum_{a \in \bar{C}} \beta(a, f) \leq 2\pi\lambda \tag{6}$$

[13]. The arguments we use to prove Theorem 1 also give

THEOREM 3 *Let f be a meromorphic function of lower order $1/2 < \lambda < \infty$ for which equality holds in (6). Then $\lambda = n/2$ for an integer $n \geq 2$ and all positive numbers $\beta(a, f)$ are equal to π.*

The following questions remain unsolved:

1. Does one really need the assumprion that the lower order is finite in Theorem 1? It seems plausible that there are no functions of infinite order with equality in (2).
2. Do the assumptions of Theorem 1 imply the regularity property (5), which is stronger then (3)?
3. Do the assumptions of Theorem 3 imply (3) or even (5)?

Our proof of Theorem 1 is based on a potential theoretic method developed in [8], [6] and [7]. We also use some notations from [5], which we recall now.

2. DELTA-SUBHARMONIC FUNCTIONS

A delta-subharmonic function is the difference of two subharmonic functions. We consider only delta-subharmonic functions in the plane C with the property $u(0) = 0$. By $\mu = \mu_u$ we denote the Riesz charge of u. We have the Jordan decomposition $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are Borel locally finite measures in the plane. We consider the space of locally finite Borel charges in the plane with weak topology: $\mu_n \to \mu$ means that

$$\int \varphi \, d\mu_n \to \int \varphi \, d\mu$$

for every continuous function φ with compact support. The non-negative charges (measures) form a cone in this space, which defines a partial order. The least upper bound $\vee$ and greatest lower bound $\wedge$ are defined for finite families of charges. We use the following notations:

$$D(r) = \{z : |z| \leq r\},$$

$$B_u(r) = \sup_\theta u(re^{i\theta}),$$

$$m_u(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u^+(re^{i\theta}) \, d\theta,$$

$$N_u(r) = \int_0^r \mu^-(D(t)) \frac{dt}{t},$$

$$T_u(r) = m_u(r) + N_u(r).$$

The Poisson-Jensen Formula in these notations takes the form $T_u(r) = T_{-u}(r)$.

For every $r > 0$ denote by $R_u(re^{i\theta})$ the even function of θ, $|\theta| \leq \pi$, decreasing for $\theta \in [0, \pi]$ and equimeasurable with $\theta \mapsto u(re^{i\theta})$. This means that the sets $\{\theta \in [-\pi, \pi] : R_u(re^{i\theta}) > x\}$ and $\{\theta \in [-\pi, \pi] : u(re^{i\theta}) > x\}$ have the same measure for all $x \in \boldsymbol{R}$. Define the Baernstein *-function [2] by

$$T_u^*(re^{i\phi}) = \frac{1}{\pi} \int_0^\phi R_u(re^{i\theta}) \, d\theta + N_u(r). \tag{7}$$

This function is subharmonic in the upper half-plane, continuous in the closed upper half-plane and has the following properties:

$$T_u^*(r) = N_u(r), \quad r > 0, \tag{8}$$

$$\frac{\partial}{\partial \phi} T_u^*(r) = \frac{1}{\pi} B_u(r), \quad r > 0, \tag{9}$$

$$\frac{\partial}{\partial \phi} T_u^*(-r) = \frac{1}{\pi} \inf_\theta u(re^{i\theta}), \quad r > 0, \tag{10}$$

$$\max_\phi T_u^*(re^{i\phi}) = T_u(r). \tag{11}$$

In addition to these properties we will use the fact that the function $r \mapsto T^*(re^{i\theta})$ is convex with respect to logarithm for every θ.

The following inequality is a slight modification of the one used by M. Sodin in [17].

LEMMA 1 *Let u be a delta-subharmonic function, such that*

$$\inf_{\theta} u(re^{i\theta}) \le 0, \quad r > 0. \tag{12}$$

Choose $0 < P < Q < \infty$ and $1/2 < \lambda < \infty$. Set $\beta = \pi/(2\lambda)$, denote by ν the Riesz measure of the subharmonic function T_u^* and put

$$S_u(r) = 2\pi \int_{D(r,\beta)} \cos(\lambda \arg z)\, d\nu,$$

where $D(r, \beta) = \{z : 0 < |z| < r, 0 < \arg z < \beta\}$. Then

$$\int_P^Q \left\{ \frac{1}{\pi} B_u(r) - \lambda T_u^*(re^{i\beta}) + \lambda S_u(r) \right\} \frac{dr}{r^{\lambda+1}} \le c \left\{ \frac{T_u(2Q)}{Q^\lambda} + \frac{T_u(2P)}{P^\lambda} \right\}, \tag{13}$$

where c is an absolute constant.

Sketch of the proof. First remark that the conditions (12) and (10) imply that T_u^* has a subharmonic extension to the whole plane. Namely we set $T_u^*(\bar{z}) = T_u^*(z)$, $\Im z > 0$. We assume that T^* is smooth in the upper half-plane. The general case is obtained using the standard approximation arguments given in [17]. Put $D = \{z : P < |z| < Q, 0 < \arg z < \beta\}$ and define $V(re^{i\theta}) = r^{-\lambda} \cos \lambda\theta$. Apply the Second Green identity

$$\int_{\partial D} \left(T_u^* \frac{\partial V}{\partial n} - V \frac{\partial T_u^*}{\partial n} \right) ds = \iint_D (T_u^* \Delta V - V \Delta T_u^*)\, d\sigma,$$

where $\partial/\partial n$ is differentiation in the direction of outer normal, ds is the length element and $d\sigma$ is the area element.

Using the property (9) of T^* and harmonicity of V we get

$$\int_P^Q \left\{ \frac{1}{\pi} B_u(r) - \lambda T_u^*(re^{i\beta}) \right\} \frac{dr}{r^{\lambda+1}}$$
$$- \int_0^\beta Q^{-\lambda} \left(\lambda T_u^*(Qe^{i\theta}) + Q \frac{\partial}{\partial r} T_u^*(Qe^{i\theta}) \right) \cos \lambda\theta\, d\theta$$

$$+\int_0^\beta P^{-\lambda}\left(\lambda T_u^*(Pe^{i\theta})+P\frac{\partial}{\partial r}T_u^*(Pe^{i\theta})\right)\cos\lambda\theta\,d\theta$$
$$=-\int_D r^{-\lambda}\cos(\lambda\arg z)\,d\nu=-\int_P^Q r^{-\lambda}\,dS(r).$$

Now we perform the following transformations of this formula:

1. Drop the third integral in the left side. (This integral is positive because $r\mapsto T_u^*(re^{i\theta})$ is a convex with respect to logarithm);
2. Drop $\cos\lambda\theta$ in the second integral in the left side (the integrand is positive).
3. Use the estimate

$$Q\frac{\partial}{\partial r}T_u^*(Qe^{i\theta})\le\frac{1}{\log 2}\int_Q^{2Q} r\frac{\partial}{\partial r}T_u^*(re^{i\theta})\,d\log r$$
$$=\frac{1}{\log 2}T_u^*(2Qe^{i\theta})\le\frac{1}{\log 2}T_u(2Q).$$

4. Integrate the right side by parts

$$-\int_P^Q r^{-\lambda}\,dS(r)\le P^{-\lambda}S(P)-\lambda\int_P^Q S(r)\frac{dr}{r^{\lambda+1}}$$

and estimate the integrated term using Jensen's formula (recall that T_u^* has been defined in the whole plane in the beginning of the proof) and (11) in the following way:

$$S(P)\le 2\pi\int_{D(P,\beta)}d\nu\le 2\pi\int_{D(P)}d\nu\le\frac{2\pi}{\log 2}\int_P^{2P}\frac{\nu(D(t))\,dt}{t}$$
$$\le 2\pi\int_0^{2P}\frac{\nu(D(t))\,dt}{t}=\int_{-\pi}^{\pi}T_u^*(2Pe^{i\theta})\le 2\pi T_u(2P).$$

Combining all these estimates we get the conclusion of the lemma.

2. *Proof of Theorem 1* We may assume without loss of generality that ∞ is not a Valiron exceptional value of f. This means

$$m(r,f)=o(T(r,f)),\quad r\to\infty \tag{14}$$

and

$$N(r,f)\sim T(r,f),\quad r\to\infty. \tag{15}$$

Fix a λ between the order and the lower order of f. Then there exist Pólya peaks of order λ. This means that there is a sequence $r_j \to \infty$ such that

$$T(r, f) \leq (1+\epsilon_j)\left(\frac{r}{r_j}\right)^{\lambda} T(r_j, f), \quad \epsilon_j r_j \leq r \leq \frac{r_j}{\epsilon_j}. \qquad (16)$$

In what follows we will select if necessary subsequences of Pólya peaks without changing the notations. Let $B(f) = \{a_k\} \subset C$. Using the theorem of Anderson-Baernstein [1], we may find a subsequence of Pólya peaks such that the following limits exist in L^1_{loc}:

$$u_k = \lim_{j\to\infty} \frac{1}{T(r_j, f)} \log \frac{1}{|f(r_j z) - a_k|}, \quad k = 1, 2, \ldots. \qquad (17)$$

The Riesz charges of delta-subharmonic functions in (17) converge weakly. The limit functions u_k is (17) are non-negative, which follows from (14). They also have disjoint supports, because the numbers a_k are distinct.

Denote by κ the measure in the plane, which counts poles of f (according to their multiplicity). In other words, $\kappa(E)$ is the number of poles of f on E. Then define $\kappa_j(E) = (T(r_j, f))^{-1}\kappa(r_j E)$. We have the weak convergence of measures

$$\kappa_j \to \mu_0. \qquad (18)$$

Similarly we define the limit function T_0 for the Nevanlinna characteristic:

$$T_0(r) = \lim_{j\to\infty} \frac{T(r_j r, f)}{T(r_j, f)}. \qquad (19)$$

It is clear from (15) that

$$T_0(r) = \int_0^r \mu_0(D(t))\frac{dt}{t}. \qquad (20)$$

We also have

$$\mu^+_{u_k} \leq \mu_0 \qquad (21)$$

and

$$T_{u_k}(r) = \int_0^r \mu^+_{u_k}(D(t))\frac{dt}{t}. \qquad (22)$$

Now from (16) we conclude that

$$T_0(r) \le r^\lambda, \quad r > 0 \quad \text{and} \quad T_0(1) = 1. \tag{23}$$

LEMMA 2 ([8]) *Let $\{u_k,\}$ be a family of non-negative delta-subharmonic functions with disjoint supports and $\mu^+_{u_k} \le \mu_0$ for some Borel measure μ_0. Then $\sum_k \mu^+_{u_k} \le 2\mu_0$.*

Using this lemma, (20), (21) and (22) we conclude that

$$\sum_k T_{u_k}(r) \le 2T_0(r), \quad r > 0. \tag{24}$$

We are going to apply the inequality (13) to functions u_k. But first we make an additional transformation in this inequality. Put $F_u(r) = T_u(r) - T^*_u(re^{i\beta}) \ge 0$, substitute $T^*_{u_k} = T_{u_k} - F_{u_k}$ in (13) and integrate by parts using (22):

$$-\int_P^Q \lambda T_{u_k}(r)\frac{dr}{r^{\lambda+1}} \ge -\lambda T_{u_k}(P)P^{-\lambda} - \int_P^Q \mu^+_{u_k}(D(r))\frac{dr}{r^{\lambda+1}}.$$

Denote in addition $G_u(r) = (1/\pi)B_u(r) - \mu^+_u(D(r))$. We obtain from (13)

$$\int_P^Q \{G_{u_k}(r) + \lambda F_{u_k}(r) + \lambda S_{u_k}(r)\}\frac{dr}{r^{\lambda+1}} \le c_1\{Q^{-\lambda}T_{u_k}(2Q) + P^{-\lambda}T_{u_k}(2P)\}. \tag{25}$$

Now from the assumption of the theorem follows that for every $\epsilon > 0$ there exists natural q such that

$$\frac{1}{\pi}\sum_{k=1}^{q} \log^+ M(r, a_k, f) \ge (2-\epsilon)A(r, f) = (2-\epsilon)rT'(r, f). \tag{26}$$

This implies

$$\frac{1}{\pi}\sum_{k=1}^{q} B_{u_k}(r) \ge (2-\epsilon)\mu_0(D(r)) \tag{27}$$

almost everywhere. To derive (27) from (26) one uses a continuity property of B_u with respect to u, see for example [9,11].

As ϵ can be chosen arbitrarily small we obtain using Lemma 2 and the definition of G_{u_k}:

$$G_0(r) := \sum_k G_{u_k}(r) = \sum_k \left\{ \frac{1}{\pi} B_{u_k}(r) - \mu^+_{u_k}(D(r)) \right\}$$
$$\geq \left\{ \sum_k \frac{1}{\pi} B_{u_k}(r) \right\} - 2\mu_0(D(r)) \geq 0. \tag{28}$$

Now we add the inequalities (25) for all k and apply (24) to the right side. We obtain

$$\int_P^Q (G_0(r) + \lambda F_0(r) + \lambda S_0(r)) \frac{dr}{r^{\lambda+1}} \leq c_1\{Q^{-\lambda}T_0(2Q) + P^{-\lambda}T_0(2P)\}, \tag{29}$$

where we use the notations

$$F_0 := \sum_k F_{u_k} \quad \text{and} \quad S_0 := \sum_k S_{u_k}.$$

Remark that all functions under the integral are non-negative and the integral with $P = 0$ and $Q = \infty$ is convergent in view of (23).

We would like to conclude that all these functions are identically equal to zero. Unfortunately this may not be the case and we may need another limit procedure similar to (17). We consider three cases.

CASE 1. $\liminf_{Q\to\infty} Q^{-\lambda}T_0(Q) > 0$. Notice that from (23) follows

$$\limsup_{Q\to\infty} Q^{-\lambda}T_0(Q) \leq 1.$$

So one can find a sequence $Q_j \to \infty$ such that the following limit functions exist (here we apply again the Anderson-Baernstein theorem).

$$w_k(z) = \lim_{j\to\infty} Q_j^{-\lambda} u_k(Q_j z) \quad j \to \infty.$$

It is clear that each of these functions w_k is not identical zero (in fact $B_{w_k}(r) > 0$ for some r). Similarly we define, choosing if necessary a subsequence of Q_j:

$$T^*_k(re^{i\phi}) = \lim_{j\to\infty} Q_j^{-\lambda} T^*_{u_k}(Q_j re^{i\phi}),$$

$$T_k(r) = \lim_{j\to\infty} Q_j^{-\lambda} T_{u_k}(Q_j r) = \sup_\phi T^*_k(re^{i\phi}),$$

$$\mu^-_k(.) = \lim_{j\to\infty} Q_j^{-\lambda} \mu^-_{u_k}(Q\,.).$$

and

$$\mu_k^+(.) = \lim_{j\to\infty} Q_j^{-\lambda}\mu_{u_k}^+(Q\,.).$$

We have $\mu_k^+ \geq \mu_{w_k}^+$ and $\mu_k^- \geq \mu_{w_k}^-$ but in general the equalities may not hold, though of course

$$\mu_k^+ - \mu_k^- = \mu_{w_k} = \mu_{w_k}^+ - \mu_{w_k}^-. \tag{30}$$

Similarly it may happen that $T_k^* \neq T_{w_k}^*$ but we still have

$$T_k^*(re^{i\phi}) = \frac{1}{\pi}\int_0^\phi R_{w_k}(re^{i\theta})\,d\theta + \int_0^r \mu_k^-(D(t))\frac{dt}{t}, \tag{31}$$

where R_{w_k} is the decreasing rearrangement of w_k on the circle $|z| = r$. Thus, as in (9) and (8)

$$\frac{\partial}{\partial\phi}T_k^*(r) = \frac{1}{\pi}B_{w_k}(r) \tag{32}$$

and

$$T_k^*(r) = \int_0^r \mu_k^-(D(t))\frac{dt}{t}. \tag{33}$$

Now we put $\Phi(r) = G_0(r) + \lambda F_0(r) + \lambda S_0(r)$ and notice that for every $M > 1$ we have

$$\int_{M^{-1}}^{M} Q_j^{-\lambda}\Phi(Q_j r)\frac{dr}{r^{\lambda+1}} = \int_{M^{-1}Q_j}^{MQ_j}\Phi(r)\frac{dr}{r^{\lambda+1}} \to 0$$

as $j \to \infty$ because the integral

$$\int_1^\infty \Phi(r)\frac{dr}{r^{\lambda+1}}$$

is convergent. We conclude that the $Q_j^{-\lambda}\Phi(Q_j r) \to 0$. Because all three summands in Φ are non-negative we conclude the following:

$$\sum_k \frac{1}{\pi}B_{w_k}(r) - \mu_k^+(D(r)) = 0, \tag{34}$$

$$T_k(r) = T_k^*(re^{i\beta}) \tag{35}$$

and

$$S_k(r) = \int_{D(r,\beta)} \cos(\lambda \arg z)\,d\nu_k = 0, \tag{36}$$

where ν_k is the Riesz measure of T_k^*.

Now we have from (27)

$$\sum_k \frac{1}{\pi} B_{w_k}(r) \geq 2\mu(D(r)), \tag{37}$$

where μ is the limit measure of μ_0 that is

$$\mu(.) = \lim_{j\to\infty} Q_j^{-\lambda} \mu_0(Q .).$$

Combining (37) with (34) and taking into account that $\sum_k \mu_k^+ \leq 2\mu$ (which follows from Lemma 2) we conclude that

$$\frac{1}{\pi} B_{w_k}(r) = \mu_k^+(D(r)), \quad k = 1, 2, \dots \tag{38}$$

and

$$\sum_k \mu_k^+ = 2\mu. \tag{39}$$

We denote by T the limit function of T_0, that is

$$T(r) = \lim_{j\to\infty} Q_j^{-\lambda} T_0(Q_j r) = \int_0^r \mu(D(t)) \frac{dt}{t}.$$

We have in view of (23)

$$T(r) \leq c_3 r^\lambda, \quad r > 0. \tag{40}$$

CASE 2. $\liminf_{P\to 0} P^{-\lambda} T_0(P) > 0$. Then we apply a similar argument using the fact that

$$\int_0^1 \Phi(r) \frac{dr}{r^{\lambda+1}}$$

is convergent. Again in this case we arrive to the limit functions w_k, T_k^*, T_k and T and limit measures μ_k^+, μ_k^- and μ with the properties (35), (36), (38) and (40).

CASE 3. $\liminf_{P\to 0} P^{-\lambda} T_0(P) = 0$ and $\liminf_{Q\to\infty} Q^{-\lambda} T_0(Q) = 0$. In this case the integral is the left side of (29) with $P = 0$ and $Q = \infty$ is convergent and non-positive. So the function under the integral is zero almost everywhere and we just take $w_k = u_k$, $T_k = T_{u_k}$, $T_k^* = T_{u_k}^*$ and so on.

In any case we get (38), (35) and (36). Let us draw the conclusions from these equations.

First from (36) follows that T_k^* are harmonic in the sector $0 < \arg z < \beta$ for all $k = 1, 2, \ldots$. Second, from (35) we conclude that $T_k^*(re^{i\beta}) = T_k(r) = \sup_\theta T_k^*(re^{i\theta})$ so, as $T_k^*(re^{i\phi})$ is increasing with respect to θ,

$$\frac{\partial}{\partial\theta}T_k^*(re^{i\beta}) = 0. \tag{41}$$

Now we want to conclude with the help of (38) that $T_k^*(r) = 0, r > 0$, which is a bit more complicated. Consider the function

$$U(z) = T_k^*(z^{\beta/\pi}) = T_k^*(z^{1/(2\lambda)}), \quad \Im z > 0,$$

where we use the branch $z^{\beta/\pi}$ which is positive for $z > 0$. Extend U to the lower half-plane by $U(z) = U(\bar{z})$. The function U, defined in this way in the whole plane, is subharmonic and its Riesz measure is supported by the positive ray. This follows from harmonicity of T_k^* for $0 < \arg z < \beta$, (41) and (32). Furthermore, it satisfies

$$U(z) \leq c|z|^{1/2}. \tag{42}$$

So we have the Weierstrass canonical representation

$$U(-r) = r\int_0^\infty \frac{n(t)\,dt}{t(t+r)}, \tag{43}$$

where $n(t) = \mu_U(D(t))$. The Riesz measure of U can be expressed in terms of B_{w_k} using (32). Namely,

$$\begin{aligned} n(r) &= \frac{1}{\pi}\int_0^r \frac{\partial}{\partial\theta}U(t)\frac{dt}{t} = \frac{\beta}{\pi^2}\int_0^r \frac{\partial}{\partial\theta}T_k^*(t^{\beta/\pi})\frac{dt}{t} \\ &= \frac{\beta}{\pi^3}\int_0^r B_{w_k}(t^{\beta/\pi})\frac{dt}{t}. \end{aligned} \tag{44}$$

Now we have

$$U(-r) = T_k^*(r^{\beta/\pi}e^{i\beta}) = T_k(r^{\beta/\pi}). \tag{45}$$

The equality (38) means $(1/\pi)B_{w_k}(r) = \mu_k^+(D(r))$ or, after integration

$$\frac{1}{\pi}\int_0^r B_{w_k}(t)\frac{dt}{t} = T_k(r). \tag{46}$$

Combining (44), (45) and (46) we get

$$\pi n(r) = U(-r). \tag{47}$$

We consider (47) together with (43) as an integral equation with respect to $n(t)$, which satisfies $0 < n(t) < c\sqrt{t}$ in view of (42). An application of Mellin or Laplace transform as in [16], [18, p. 305] shows that the only solutions are $n(t) = C\sqrt{t}$. Now the evaluation of the integral in (43) shows that $U(r) = 0$, $r > 0$. Recalling the definition of U, we obtain that $T_k^*(r) = 0$, $r > 0$, which implies (see (33)) that $\mu_k^- = 0$. In particular, w_k are subharmonic and

$$\mu_k^+ = \mu_{w_k}^+ = \mu_{w_k}$$

in view of (30). Inequality (21) implies $\mu_k^+ \leq \mu$, thus $\mu_{w_k} \leq \mu$. So we have from (39)

$$\sum_k \mu_{w_k} = 2\mu \geq 2 \bigvee_k \mu_{w_k}. \tag{48}$$

The functions w_k are non-negative and have disjoint supports. This follows from the corresponding properties of u_k. Now from (40), (24) and the definition of T and w_k follows

$$\sum_k w_k(z) \leq C|z|^\lambda. \tag{49}$$

So from the subharmonic version of the Denjoy-Carleman-Ahlfors theorem (see for example [11]) follows that the number of functions w_k is finite, namely at most 2λ. Denote this number by n.

Now we are in position to use the following

LEMMA 3 *Let $w_1, \ldots, w_n$ be non-negative subharmonic functions in the plane with disjoint supports, satisfying (49) and (48). Then $2\lambda = n \geq 2$ and*

$$\sum_{k=1}^{q} w_k(re^{i\theta}) = Cr^\lambda |\cos\lambda(\theta + \theta_0)|$$

with some constants $C > 0$ and $|\theta_0| \leq \pi$.

This lemma was first proved in [6], see also [7] for a slight generalization.

Now we can finish the proof of the theorem. First we conclude that only half-integral orders of Polya peaks λ are possible. On the other hand it is known that possible orders of Polya peaks fill the interval between the lower order and the order. So we conclude that this interval is reduced to a point, which gives (3). Then we conclude from the lemma that $b(a, f) \leq 2\pi/n$, $a \in B(f)$ and card$B(f) = n$. So if we have equality in (2) then $b(a, f) = 2\pi/n$, $a \in B(f)$. This proves the theorem.

Remark The proof of Lemma 3 in [6], [7] is quite technical. We can simplify the argument using the result of M. Essen and D. Shea [10], which states that if w is a subharmonic function and

$$T^*_w(re^{i\theta}) = Cr^\lambda \sin\lambda\theta, \quad 0 < \theta < \pi/(2\lambda)$$

then the support of w is a sector of the form $|\arg z - \theta_0| < \pi/(2\lambda)$ and w is positive and harmonic in this sector. As soon as we know that our functions w_k have such form, Lemma 3 becomes almost trivial.

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