Julia sets are uniformly perfect

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According to Pommerenke [1,2] a compact set $E \subset \bar{\mathbf{C}}$ is called uniformly perfect if the doubly connected domains in $\bar{\mathbf{C}} \backslash E$ that separate E have bounded moduli. The limit sets of finitely generated non-elementary Kleinian groups as well as Julia sets of hyperbolic rational functions are known to have this property [2]. In [3] Hinkkanen proves that Julia sets of all polynomials are uniformly perfect. Recently he extended his proof to the case of rational functions [4]. In this note we give a simple proof of Hinkkanen's theorem for arbitrary rational functions.

Let N be an open set in $\bar{\mathbf{C}}$. We assume that each component of N is hyperbolic. A closed curve in N is called trivial if it is freely homotopic to a point in N. A closed curve in N is called peripheric if it is either trivial or homotopic to an arbitrarily small loop around a puncture of N. $l_N(C)$ stands for the Poincaré length of a curve $C \subset N$.

Proposition 1 *If the Poincaré lengths of all non-trivial closed curves in N are bounded from below by a constant $\delta > 0$ then $E = \bar{\mathbf{C}} \backslash N$ is uniformly perfect.*

Proof. Let $A \subset N$ be a non-degenerate annulus separating E. Denote by C the simple closed curve separating ∂A and having minimal Poincaré length in A. Then C is non-trivial in N. By the generalized Schwarz lemma we have $l_A(C) \geq l_N(C)$. So

$$\operatorname{mod}(A) = \frac{\pi}{l_A(C)} \leq \frac{\pi}{l_N(C)} \leq \frac{\pi}{\delta}.$$

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QED.

Remark. The converse is also true. Indeed, assume that there is a simple closed Poincaré geodesic $\gamma \subset N$ having length l. Then by the Collar lemma [5] there exists an annulus A embedded in N, such that γ is non-trivial in A, and the hyperbolic area of A is $2l/\sinh(l/2)$. Then $\mathrm{mod}(A) \geq (l\sinh(l/2))^{-1} \to \infty,\ l \to 0$.

Now we prove that the set of normality $N(f)$ (the complement of the Julia set) of a rational function f, $\deg f \geq 2$ satisfies the assumption of the Proposition. Let γ be a closed non-trivial curve in $N(f)$. The description of the dynamics of f on the set $N(f)$ due to Fatou and Sullivan [6-8] implies that one of the following statements is true:

a). $f^n(\gamma)$ is trivial in $N(f)$ for some $n \in \mathbf{N}$ or

b). $f^n(\gamma)$ separates the boundary of Arnold – Herman ring for some $n \in \mathbf{N} \cup \{0\}$.

In the case b) we are done because: $l_N(\gamma) \geq l_N(f^n\gamma)$, there are finitely many Arnold – Herman rings, and they are all non-degenerate.

In the case a) take n as small as possible and denote by D the component of $N(f)$ containing $f^n\gamma$. Then $f^n\gamma$ is non-peripheric in $D\backslash\{$critical values of $f\}$. (Otherwise we could deform $f^n\gamma$ to a point in D or to a small loop around a critical value without hitting critical values, so $f^{n-1}\gamma$ would be trivial in $N(f)$). Choose three points $b_i \in J = \bar{\mathbf{C}}\backslash N(f)$ and set $G = \bar{\mathbf{C}}\backslash\{b_1, b_2, b_3\}$. Then $f^n\gamma$ is trivial in G but non-peripheric in $G\backslash\{$critical values of $f\}$, so

$$l_N(\gamma) \geq l_N(f^n\gamma) \geq l_G(f^n\gamma) \geq \delta,$$

where δ is the minimal hyperbolic distance between critical values of f with respect to G. QED.

Remark. If f is an entire transcendental function then $N(f)$ can have doubly connected components of arbitrarily large moduli [9].

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