On may 7 we wrote:

"Sasha,

Let me try to make more precise the questions which we are asking you.

1. Let $(\omega_1, \omega_2) = (1, \tau)$ be an *arbitrary* lattice. $\Im\tau > 0$. Consider non-zero meromorphic functions satisfying

$$f(z + 2\omega_1) = k_1(f)f(z), \quad f(z + 2\omega_2) = k_2(f)f(z), \tag{1}$$

with some constants $k_1, k_2 \in \mathbf{C}^*$, constants depending on f. Proportional functions are considered as equivalent.

a) Every such function has d zeros, d poles, and $2d$ critical points per period, with some $d \geq 1$; we call it the *degree*. Indeed, $g = f'/f$ is doubly periodic, with all poles simple. The residues of g at zeros of f are 1 and the residues of g at poles of f are -1. And zeros of g are critical points of f.

b) There exists a function f with arbitrarily prescribed zeros and poles. Just take a ratio of products of sigma functions. As the numbers of factors in the numerator and denominator are equal, the ratio will satisfy (1).

c) Two functions f with same zeros and poles differ by an exponential factor e^{qz}. We can always choose this q so that $k_1 = 1$.

Thus the set of functions with the property (1) with $k_1 = 1$ up to proportionality, depends on $2d$ parameters, zeros and poles.

We want to find f of degree d with prescribed critical points c_j. We have $2d$ equations saying that the critical points of f are prescribed, and $2d$ unknowns, parameters of f.

Question 1. How many solutions such a system has, for generic critical points?

We feel that you know something about this question. Bethe ansatz equations for it can be written in various ways.

This is a correct analog of the question "how many rational functions exist with prescribed generic critical points", to which the answer is Catalan number."

ADDED on May 8. Here is a computation for $d = 1$. I use the standard notation (Akhiezer, Whittaker–Watson). Let

$$f(z) = e^{qz}\frac{\sigma(z-a)}{\sigma(z-b)}.$$

We have $\sigma(z+2\omega_j) = e^{2\eta_j(z+\omega_j)}\sigma(z)$, so

$$f(z+2\omega_j) = \exp\left(2q\omega_j + 2\eta_j(b-a)\right) f(z).$$

Our condition that ω_1 is the exact period gives

$$q\omega_1 + \eta_1(b-a) = \pi i n, \tag{2}$$

with some integer n. Taking logarithmic derivatives, we get

$$f'(z)/f(z) = q + \zeta(c_j - a) - \zeta(c_j - b) = 0, \quad j = 1, 2, \tag{3}$$

where c_j are the prescribed critical points. This is an elliptic function, therefore $a + b \equiv c_1 + c_2$. Shifting the origin to the point $c_1 + c_2$, we may assume $c_1 + c_2 = 0$, $c := c_1 = -c_2$, and $b = -a$. Solving (2) for q and substituting to (3), we obtain

$$\pi i n + 2\eta_1 a + \omega_1\left(\zeta(c-a) - \zeta(c+a)\right) = 0.$$

Let us write these equations as

$$g(a) = \pi i n,$$

where

$$g(a) = \omega_1(\zeta(c-a) - \zeta(c+a)) + 2\eta_1 a.$$

Let us fix some period parallelogram Q. It is clear that for every sufficiently large n, the equation $g(a) = \pi i n$ has one solution $a_n \in Q$ near the pole of $g(c-z)$. This implies that the answer to Question 1 is infinity.

Remark.

Let a be a solution for some n. Then $a + 2\omega_1$ is a solution with the same n, while $a+2\omega_2$ is a solution with $n' = n-2$. This follows from the properties of ζ,

$$\zeta(z+2\omega_j) = \zeta(z) + \eta_j, \quad j = 1, 2,$$

and from Legendre's relation

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}.$$

To obtain finitely may solutions, we modify the question as follows.

Question 2. How many proportionality classes of f with the properties (1) with $k_1 = 1$ and $|k_2| = 1$ exist for prescribed critical points?