Some developments arising from Picard's theorems

Alex Eremenko

Purdue University

eremenko@purdue.edu

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Emile Picard (1879):
If an entire function omits two values, then it is constant.
I recall that a function is entire if it is analytic in the whole complex plane, or alternatively, if it is represented by an everywhere convergent power series.
This theorem (and its two-lines proof) made an enormous impression, and was subject to many generalizations.
Meromorphic functions are ratios of entire ones; at the points where the denominator is zero they take infinite values, so they are mapping the complex plane into the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
We can restate Picard's theorem as: If a meromorphic function in the plane omits three values then it is constant. The generalization known as the Great Picard's theorem says: If a meromorphic function in a punctured neighborhood of a point omits three values in $\widehat{\mathbb{C}}$ then this point is a pole or a removable singularity.

Schottky (1904) proved the corresponding result for the unit disk. The family of all meromorphic functions in the unit disk which omit three values, say $0, 1, \infty$ is normal.

A family F of meromorphic functions in a region D is called normal if from any sequence (f_n) in F one can select a subsequence which converges uniformly (with respect to the spherical metric in the image) on every compact subset of D.

Picard and Montel theorems can be simply derived from each other, and this is the basis of the famous

Bloch's Principle: If some conditions imply that a meromorphic function in the plane is constant, then the same conditions imposed on a family of meromorphic functions in any region imply that this family is normal, and conversely.

The condition that some value a is omitted can be replaced by the condition that all solutions of $f(z) = a$ have high multiplicity. Nevanlinna proved the following generalization of Picard's theorem: If f is a non-constant meromorphic function in $\mathbb{C}$, and a_j have the property that all solutions of $f(z) = a_j$ have multiplicities at least m_j, then

$$\sum_j \left(1 - \frac{1}{m_j}\right) \leq 2, \tag{1}$$

where $m_j = \infty$ if a_j is omitted. This result was obtained as a consequence of *Nevanlinna theory* which can be considered as a far-reaching quantification of Picard's theorems.

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, and

$$A(r, f) = \frac{1}{\pi} \int \int_{x^2+y^2 \leq r^2} \frac{|f'(x+iy)|^2}{(1+|f(x+iy)|^2)^2} dxdy,$$

the area of the image of the disk under f with respect to the spherical metric, divided by the area of the sphere. So for a rational function of degree d we have $A(r, f) \to d, r \to \infty$. Then the Nevanlinna characteristic is defined as an average of $A(r, f)$ with respect to r:

$$T(r, f) = \int_0^r \frac{A(t, f)}{t} dt.$$

This function measures the growth of a meromorphic function. For a rational function $T(r, f) \sim d \log r$, and for transcendental meromorphic functions we have $T(r, f)/\log r \to \infty$. Nevanlinna characteristic has formal algebraic properties similar to the degree of a rational function, and this is one reason why we average with respect to r.

The order of a meromorphic function is defined as

$$\rho := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \in [0,\infty].$$

So for example, e^z and $\sin z$ are of order 1, e^{z^p} is of order p, $\wp(z)$ has order 2, and e^{e^z} is of infinite order.
For every $a \in \widehat{\mathbb{C}}$ we define $n(r,a,f)$ as the number (counting multiplicity) of solutions of $f(z) = a$ in the disk $|z| \leq r$, and then apply the averaging in r as above:

$$N(r,a,f) = \int_0^r (n(t,a,f) - n(0,a,f))\frac{dt}{t} + n(0,a,f)\log r.$$

Then the First Main Theorem of Nevanlinna says that for all $a \in \widehat{\mathbb{C}}$ we have

$$N(r,a,f) \leq T(r,f) + O(1), \quad r \to \infty,$$

and the Second Main Theorem says that for any finite set $\{a_1, \ldots, a_q\}$

$$\sum_{j=1}^{q} N(r, a_j, f) \geq (q - 2 + o(1))\, T(r, f) + N_1(r, f),$$

when $r \to \infty$ avoiding a set of finite measure. Here $N_1(r, f)$ is the ramification term: $n_1(r, f)$ is the number of critical points (counting multiplicity) in the disk $|z| \leq r$, and N_1 is obtained from n_1 by the averaging as above.

The meaning of the Second Main Theorem can be make clearer if we define the deficiency

$$\delta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, a, f)}{T(r, f)}.$$

So $0 \leq \delta(a, f) \leq 1$, and if the value a is omitted then $\delta(a, f) = 1$. Then the Second Main Theorem implies that $\delta(a, f) > 0$ for at most countable set of a (they are called deficient), and

$$\sum_{a \in \widehat{\mathbb{C}}} \delta(a, f) \leq 2. \qquad (2)$$

This is a generalization of Picard's theorem. If one takes multiplicities into account, then (1) follows.

Returning to Picard's theorem, it inspired a lot of variations and generalizations in 20th century. Our main subject here will be the analogs of Picard's theorem involving derivatives.

Among the earliest results on this subject, I mention a theorem of Milloux (1940) If an entire function omits some finite value, and its derivative omits some non-zero value, then f is constant. The corresponding normality criterion was proved by Miranda (1935). Example $e^z + c$ shows that f' can indeed omit zero. This shows that the value 0 is special for the derivative.
On the other hand, Csillag (1935) proved that If f is entire and $f, f^{(m)}, f^{(n)}$ omit zero for some $0 < m < n$, then $f(z) = e^{az+b}$. The case $(m, n) = (1, 2)$ was earlier proved by Saxer.

A great breakthrough in these questions was made by Hayman (1959).

Hayman's Alternative. If f is a transcendental meromorphic function in the plane and n is a positive integer, then either f assumes every finite value infinitely many times, or $f^{(n)}$ assumes every finite non-zero value infinitely many times.

It is noteworthy that this theorem gives only two conditions on omitted values which imply that the function it is constant. Hayman's result generalizes the theorem of Milloux to meromorphic functions.

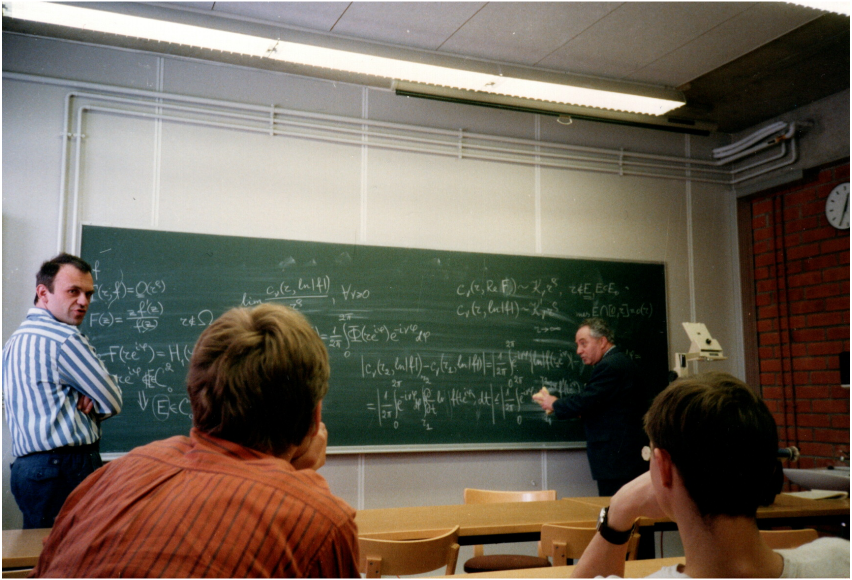
In this paper Hayman proved many other results, but he assumed that they are not best possible, and stated several conjectures. The most outstanding conjectures are these two:

Hayman's Conjecture 1. If f is a transcendental meromorphic function in $\mathbb{C}$, and $n \geq 2$, then $(f^n)'$ assumes all non-zero complex values infinitely many times.
Hayman proved this for $n \geq 4$, and later Mues extended this to $n = 3$, and Clunie proved this conjecture for entire functions.

Hayman's conjecture 2. If f is a meromorphic function in $\mathbb{C}$ and both f and $f^{(n)}$ are free of zeros, then $f(z) = e^{az+b}$ or $f(z) = (az + b)^{-m}$ with some integer n.
This was proved by Frank for $n \geq 3$ and by Mues for functions of finite order. The final result with $n = 2$ is due to Langley (1993) who developed the methods of Mues and Frank in a very subtle way.

Walter in Joensuu

I met Walter in winter 1992/3 in a conference in Joensuu. For me this was probably the first international conference that I was able to attend (I moved from the Soviet Union to the US in 1991). We shared a 3-bedroom apartment with my adviser A. A. Goldberg and for him this was also one of the first trips abroad. We had no common language for all three of us: Goldberg did not speak English, so he had to communicate with Walter in German and with me in Russian. And I translated his conference talk into English.



I am translating Goldberg's talk, Walter is listening

Next summer I was invited by Walter to stay with him and his wife in Aachen, where he had his first position. When meeting me in the train station, he explained to me his proposal to prove Hayman's Conjecture 1 in the remaining case $n = 2$ for functions of finite order.

The result was the paper which we wrote that summer and submitted to Revista Math Iberoamericana. The proof was unusual for the theory of meromorphic functions since it used holomorphic dynamics. Holomorphic dynamics which was originated by Fatou (with some results obtained independently by Julia, Lattès and Ritt) in the beginning of 20th century experienced a strong revival since 1982.

We were both interested in holomorphic dynamics at that time: I was deeply impressed with the work of Misha Lyubich in the early 1980s, and we wrote several joint papers in Kharkiv on dynamics of transcendental functions. Walter started thinking about holomorphic dynamics under the influence of J. Hubbard, when he visited Cornell University as a postdoc in 1987-89.

Walter and I in Aachen in 1993

The idea that Walter explained to me in the train station: if f has finitely many zeros, then Hayman's alternative is applicable, and we are done. So assume infinitely many zeros. They are all multiple zeros of f^n. Then the function

$$g(z) = z - f^n$$

has infinitely many fixed points, and $g'(a) = 1$ at all these fixed points a. To each such fixed point an immediate domain of attraction P is associated. This domain is invariant under g, and according to the fundamental theorem of Fatou, P must contain either critical or asymptotic value of g. If this singular value which exists by Fatou's theorem is critical then $g'(z) = 0$ at some point z in the domain of attraction of P, and since these domains are disjoint for different neutral fixed points, we obtain infinitely many critical points, which are solutions of $(f^n)' = 1$.

So it remains to deal with the possibility that g has infinitely many asymptotic values but only finitely many critical values. Such meromorphic functions exist, an example can be found in the book by Volkovyskii.
But it turns out that such thing is impossible for functions of *finite order*. This was a great insight of Walter which he explained to me during that conversation in the train station. It took few weeks to work out the details, and the result was the following

Theorem 1. Let f be a meromorphic function of finite order ρ. Then every asymptotic value of f, except at most 2ρ of them, is a limit point of critical values different from this asymptotic value.

By using this and the dynamics argument outlined above, we obtain

Theorem 2. Let f be a transcendental meromorphic function of finite order, and assume that it has infinitely many multiple zeros. Then f' assumes every finite non-zero value infinitely often.

Unfortunately, this theorem fails for functions of infinite order, as we demonstrated by an example, so it seemed that we could not prove Hayman's Conjecture 1 in full generality with this method.

I knew from the work of Zalcman (1975) and Brody that there exists a marvelous device to reduce Picard-type theorems to their special cases for functions of finite order. This is called the Zalcman Rescaling Lemma to the specialists in classical function theory, and Brody's Rescaling Lemma to the specialists in holomorphic curves. Actually this argument goes back to the work of Bloch and Valiron in the beginning of 20th century. However this rescaling argument did not seem to be appropriate for working with derivatives.

Only next summer (1994) when browsing the book by Schiff on normal families, I learned about the paper by X. Pang of 1989 where a suitable generalization of Zalcman's Lemma was proved[1]. I immediately informed Walter (and Zalcman) and since our paper was not yet published, we managed to revise it, and obtain the full proof of Hayman's Conjecture 1 (1995).

[1] Independently of Pang, similar arguments were used in Schwick's thesis (1989)

Our final result was

Theorem 3. For every transcendental meromorphic function f, and every positive integers $k < n$, the function $(f^n)^{(k)}$ takes every finite non-zero value infinitely often.

Meanwhile, the statement of our Theorem 2 reached China (this was before the advent of the arXiv), and the result was an "independent" proof of Hayman's conjecture published by Huaihui Chen and Mingliang in 1995.
In a later paper [**?**], Walter applied Pang's lemma to give a new simple proof of Langley's theorem stated above.

Since then our Theorem 3 was very much used and generalized; our paper has 230 citations on MSN and 555 on Google Scholar at the time when I write this.

The ultimate generalization for the first derivative was conjectured by Walter and proved by Jianming Chang (2012), who used a deep intermediate result of Nevo, Pang and Zalcman (2007):

Suppose that for a transcendental meromorphic function f, $|f'|$ is bounded on the zero set of f. Then f takes every finite non-zero value infinitely often.

Despite many variations and generalizations, no new proof of Theorem 3 was found until 2020: all variants and generalizations used our Theorem 3. I always felt that our proof was unsatisfactory from the philosophical point of view. One reason for this was the use of dynamics: why a non-dynamical theorem should use dynamics in its proof? Walter showed me how dynamics can be eliminated. The argument uses instead the so-called Logarithmic Change of the Variable, which is a common tool in transcendental dynamics, but formally does not belong to it. (It was invented by Teichmüller for a problem of value distribution theory).
More important philosophical objection is related to the weird overall scheme: we prove Theorem 2 for functions of finite order, (*and it is not true without this restriction!*). And then by some magic trick we derive Theorem 3 which applies to all meromorphic functions.

The new proof, which uses neither our arguments nor Pang's lemma, was obtained recently by Ta Thi Hoai An and Nguen Viet, as a corollary of the very deep result of Yamanoi (2013), and I briefly describe his result.

Erwin Mues conjectured in 1971 that the defect relation for the derivative of a meromorphic function can be improved to

$$\sum_{a\neq\infty} \delta(a, f') \leq 1.$$

In his attempts to prove this conjecture Goldberg made a new one

For all meromorphic functions

$$\overline{N}(r, \infty, f) \leq N(r, 0, f'') + o(T(r, f)), \tag{3}$$

(as $r \to \infty$ avoiding a small exceptional set), where $\overline{N}$ is the averaged counting function of distinct poles (without multiplicity)).

Goldberg showed that this implies Mues's conjecture, and proved (3) for the case when all poles of f are simple.

It is amazing that no perturbation argument seems to derive the Mues and Goldberg conjectures from this special "generic" case.

Langley proved that for meromorphic functions of finite order the condition that $f^{(k)}$ for some $k \geq 2$ has finitely many zeros implies that f has finitely many poles. However he constructed an example of a meromorphic function of arbitrarily slow infinite order of growth for which f'' has no zeros but f has infinitely many poles.

So there is no Picard-type theorem corresponding to Goldberg's conjecture!

Goldberg's conjecture was proved by Katsutoshi Yamanoi in the following stronger form

$$k\overline{N}(r,\infty,f) + \sum_{a\in A} N_1(r,a,f) \leq N(r,0,f^{(k+1)}) + o(T(r,f))$$

outside a set of r of zero logarithmic density, where k is any positive integer and A is any finite set. This 77 pages paper is a real tour de force; this is probably the most difficult of all results in value distribution of meromorphic functions in dimension one.

In the recent decades, the main interests in Nevanlinna theory shifted from meromorphic functions to holomorphic curves, but this result of Yamanoi (and his other results) on one-dimensional Nevanlinna theory show that it is still very much alive.

Returning to Walter's work, I would like to mention here two of his other contributions.
In his paper of 1995 he proved the following

Theorem 4. Let f be a meromorphic function of finite order, and $a \in \mathbb{C}$. If $a \neq 1$ and $a \neq (n+1)/n$ for all $n \in \mathbb{N}$, then $ff'' - a(f')^2$ has infinitely many zeros, unless $f = Re^P$ with rational R and polynomial P.

He conjectured that the assumption of finite order can be removed in this theorem; this would give a generalization of Langley's proof of Hayman's Conjecture 2, which corresponds to $a = 0$. Walter's conjecture was proved by Langley (1996), without use of Pang's lemma.

In their joint paper of 2005, Walter and Jim study the question whether in Hayman's alternative one can replace omitted values by high multiplicity. As expected, they show that this is so, but probably their results are not best possible. Improving a previous result by Zhen Hua Chen, they show that a meromorphic function f whose zeros have multiplicity $\geq m$ while 1-points of $f^{(k)}$ have multiplicity $\geq n$ must be constant if

$$\left(2k+3+\frac{2}{k}\right)/m+\left(2k+2+\frac{2}{k}\right)/n<1.$$

So when $k=1$, we have the condition

$$\frac{7}{m}+\frac{8}{n}<1.$$

They also constructed an example of a non-constant meromorphic function with $m=2$ and $n=3$.

On the other hand, Nevo, Pang and Zalcman proved that

The derivative of a transcendental meromorphic function whose almost all zeros are multiple takes every non-zero value infinitely often.

Notice that the rational function

$$f(z) = \frac{(z-a)^2}{(z-b)}$$

has only one zero in $\mathbb{C}$, this zero is multiple, and f' omits 1. So it is an open problem to obtain a best possible result about multiplicities in Hayman's alternative. The statements about normality corresponding to this via Bloch's Principle also hold in this setting.