Spherical metrics with conic singularities

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Consider two polygons in the plane, and let f be a conformal map of one onto another sending vertices to vertices. Suppose that interior angles at the corresponding vertices are equal. Then $f(z) = az + b$.

To prove this statement, consider the Schwarz–Christoffel maps from the upper half-plane onto our polygons. Since the polygons are conformally equivalent both maps must be of the form

$$f_j(z) = C_j \int_{z_0}^{z} \prod_{k=1}^{n} (\zeta - a_k)^{\alpha_k - 1} d\zeta + C'_j, \quad j \in \{1, 2\}, \tag{1}$$

with the same sequences a_k and α_k, and the statement immediately follows.

Christoffel–Schwarz formula also proves the existence of a polygon with given angles and in given conformal class, provided that the following relation for the angles holds

$$\sum_{k=1}^{n} \alpha_k = n - 2. \tag{2}$$

This condition ensures that $f_j(\infty)$ in (1) is not a corner. Notice that α_k can be arbitrarily large, so our polygons are not always subsets of the plane.

By gluing a polygon to its reflected copy isometrically along the sides, we obtain a sphere with Euclidean metric with conic singularities. A generalization of the previous statement is the following:

For any given points $a_1, \ldots, a_n$ on the Riemann sphere, and any positive numbers $\alpha_1, \ldots, \alpha_n$ satisfying (2) there exists a conformal metric of zero curvature on $\overline{\mathbf{C}} \backslash \{a_1, \ldots, a_n\}$ with conic singularities at a_j with angles $2\pi\alpha_j$. This metric is unique up to scaling.

This is also true for any compact Riemann surface S, if one replaces 2 in (2) by the Euler characteristic $\chi(S)$.

Similar statement holds for metrics of constant negative curvature on any Riemann surface:

Theorem. (Picard) *For any given points $a_1, \ldots, a_n$ on a compact Riemann surface S, and any positive numbers $\alpha_1, \ldots, \alpha_n$ satisfying*

$$\chi(S) + \sum_{k=1}^{n} (\alpha_j - 1) < 0, \tag{3}$$

there exists a conformal metric of constant curvature -1 on $S \backslash \{a_1, \ldots, a_n\}$ with conic singularities at a_j with angles $2\pi\alpha_j$. This metric is unique.

Notice that Picard's theorem implies the Uniformization theorem for compact Riemann surfaces, which corresponds to $n = 0$, and uniformization theorem for 2-dimensional orbifolds, which corresponds to integer α_j.

In this lecture we consider metrics of constant *positive* curvature with conic singularities on compact surfaces, for which the corresponding questions are wide open. Only for small angles we have a complete result:

Theorem. (Feng Luo and Gang Tian [8]) *A metric of curvature* 1 *on the sphere with prescribed singularities a_k and angles $2\pi\alpha_k$, $\alpha_k \in (0, 1)$ exists if and only if*

$$0 < 2 + \sum_{k=1}^{n} (\alpha_j - 1) \leq 2 \min_j \{\alpha_j\},$$

and such a metric is unique.

A necessary condition on the angles which corresponds to the Gauss–Bonnet theorem, in the case of positive curvature reads:

$$\chi(S) + \sum_{k=1}^{n} (\alpha_j - 1) > 0. \tag{4}$$

For an n-tuple of positive numbers

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$$

we denote by $\mathrm{Sph}_{g,n}(\boldsymbol{\alpha})$ the set of Riemannian metrics of curvature 1 on a surface of genus g with n conic singularities with angles $2\pi\alpha_j$. (Conformal structure is not prescribed!) A natural topology on $\mathrm{Sph}_{g,n}(\boldsymbol{\alpha})$ is given by bi-Lipschitz distance.

Theorem 1. (Mondello and Panov [10]) *For $g \geq 1$, $\mathrm{Sph}_{g,n}(\boldsymbol{\alpha}) \neq \emptyset$ if and only if (4) holds.*

The situation for the sphere is much more complicated:

Theorem 2. (Mondello and Panov [9]) *If $\mathrm{Sph}_{0,n}(\boldsymbol{\alpha}) \neq \emptyset$, then (4) holds, and in addition*

$$d_1(\boldsymbol{\alpha} - \mathbf{1}, \mathbf{Z}_o^n) \geq 1, \tag{5}$$

where $\mathbf{1} = (\mathbf{1}, \ldots, \mathbf{1})$, d_1 is the ℓ_1 distance, and $\mathbf{Z}_o$ is the odd lattice, the set of vectors in $\mathbf{Z}^n$ with odd sum of coordinates.

Conversely, if (4) and the strict inequality in (5) hold, then $\mathrm{Sph}_{0,n}(\boldsymbol{\alpha}) \neq \emptyset$.

Theorem 3. (Eremenko [4]) *Suppose that (4) holds, and that we have equality in (5). Then $\mathrm{Sph}_{0,n}(\boldsymbol{\alpha}) \neq \emptyset$ if and only if the following conditions are satisfied:*

Suppose that $\alpha_{m+1}, \ldots, a_n$ are integers while $\alpha_1, \ldots, \alpha_m$ are not.

a) There exists a choice of signs $\epsilon_j \in \{\pm 1\}$ and an integer $k' \geq 0$ such that

$$\sum_{j=1}^{m} \epsilon_j \alpha_j = k', \tag{6}$$

and the number

$$k'' := \sum_{j=m+1}^{n} \alpha_j - n - k' + 2 \quad \text{is non-negative and even.} \tag{7}$$

b) Let

$$\mathbf{c} := (\alpha_1, \ldots, \alpha_m, 1, \ldots, 1), \quad \text{where } 1 \text{ is repeated } k' + k'' \text{ times.}$$

If $\mathbf{c} = \eta \mathbf{b}$ where where coordinates of $\mathbf{b}$ are integers with greatest common factor 1, then

$$2 \max_{m+1 \leq j \leq n} \alpha_j \leq \sum_{j=1}^{q} b_j, \quad \text{where} \quad q = k' + k'' + m.$$

(If the coordinates of $\mathbf{c}$ *are incommensurable, then condition b) is void).*

Example 1. If S is the sphere, and all α_j are integers, then the necessary and sufficient condition for existence of the metric is

$$2d-2 := \sum_{j=1}^{n}(\alpha_j - 1) \quad \text{is even, and} \quad \max_j \alpha_j \leq d.$$

So we know when $\mathrm{Sph}_{g,n}(\boldsymbol{\alpha}) \neq \emptyset$. Since every metric defines a conformal structure, we have the *forgetful map*

$$F : \mathrm{Sph}_{g,n}(\boldsymbol{\alpha}) \to \mathrm{Mod}_{g,n}$$

where $\mathrm{Mod}_{g,n}$ is the moduli space of conformal structures on a surface of genus g with n punctures.

In terms of this forgetful map, our main question is what is its image and valence. We list some general results and conjectures. Consider the set of real numbers

$$\mathrm{Crit}_{n,\boldsymbol{\alpha}} = \{\|\boldsymbol{\alpha}_I\|_1 - \|\boldsymbol{\alpha}_{cI}\|_1 + 2b : I \subset \{1,\ldots,n\},\ b \in \mathbf{Z}_{\geq 0}\}$$

and define the *non-bubbling parameter*

$$NB_{g,n,\boldsymbol{\alpha}} = d_{\mathbf{R}}(\chi(S\backslash A), \mathrm{Crit}_{n,\boldsymbol{\alpha}}),$$

where A is a set of n points, and $d_{\mathbf{R}}$ is the distance on the real line.

Theorem 4. (Mondello and Panov [10]) *If* $NB_{g,n,\boldsymbol{\alpha}} > 0$ *then the forgetful map is proper.*

This means that metrics cannot degenerate unless the conformal structure degenerates. Without the non-bubbling condition, such a degeneration is possible.

Under the condition $NB_{g,n,\boldsymbol{\alpha}} > 0$, the forgetful map has a degree, and this degree has been computed by Chen and Lin. In particular, the forgetful map is finite-to-one when $NB > 0$.

Chen and Lin use the generating function

$$g(x) = (1 + x + x^2 + \ldots)^{-\chi(S)+n} \prod_{j=1}^{n}(1 - x^{\alpha_j}).$$

Suppose that

$$g(x) = 1 + b_1 x^{n_1} + b_2 x^{n_2} + \ldots + b_k x^{n_k} + \ldots,$$

(this defines the numbers b_k).

Theorem 5. (Chen and Lin [2]) *Let us define the integer k by*

$$2n_k < \chi(S) + \sum_{j=1}^{n} (\alpha_j - 1) < 2n_{k+1}$$

this is well defined when $NB_{g,n,\boldsymbol{\alpha}} > 0$). Then the degree of the forgetful map is

$$\sum_{j=0}^{k} b_j.$$

Example 2. For a torus with one singularity with angle $2\pi\alpha$, the degree is defined when α is not an odd integer. It is equal to m where $2m$ is the closest even integer to α. When $\alpha = 2m$, then the forgetful map has m preimages for every generic point. In fact the forgetful map is complex analytic in this case, so the number of preimages is equal to the degree.

Three main methods are used to study metrics of constant positive curvature with conic singularities:

a) Synthetic geometry (partition of the surface into geodesic triangles, application of spherical trigonometry etc.)

b) The direct study of the non-linear PDE

$$\Delta u + e^{2u} = 2\pi \sum_{j=1}^{n} (\alpha_j - 1)\delta_{a_j},$$

where $ds = e^u |dz|$ is the line element of the metric in a conformal local coordinate.

c) The study of the linear Fuchsian ODEs associated with the problem.

We explain the third method. A small smooth piece of a surface of constant curvature 1 is isometric to a region on the standard sphere, this isomorphism is complex analytic, so by an analytic continuation we obtain a (multi-valued) *developing map*

$$f : S \backslash A \to \overline{\mathbf{C}},$$

where A is the set of singular points. The monodromy of this map consists of rotations of the sphere, so the Schwarzian derivative

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = R(z) \tag{8}$$

is single valued. This $R(z)$ is actually a quadratic differential on S. Local behavior at a singularity implies that all poles of R are double with principal parts

$$\frac{1-\alpha^2}{2(z-a)^2},$$

where $2\pi\alpha$ is the angle at the singularity a. To determine R completely one has to know the residues of $R(z)dz$ which are called *accessory parameters*. They should be found from the condition that the monodromy of f belongs to $PSU(2) \sim O(3)$.

The differential equation (8) is equivalent to a linear differential equation: setting $f = w_1/w_2$ we obtain that w_j are two linearly independent solutions of

$$w'' + \frac{R}{2}w = 0. \tag{9}$$

So to find the metrics of curvature 1 with prescribed angles $2\pi\alpha_j$ at the singularities a_j one has to solve the accessory parameter problem: *to determine the accessory parameters from the condition that the projective monodromy of (9) is* unitarizable *that is conjugate to a subgroup of* $PSU(2)$.

This problem reminds the early approach of Klein and Poincaré to the uniformization theorem: in a second order Fuchsian equation with prescribed principal parts at the poles, the problem was to determine accessory parameters so that the monodromy group is conjugate to a subgroup of $PSL(2, \mathbf{R})$.

In general, the preimage of the forgetful map can be infinite. This occurs when the monodromy of the developing map is a subgroup of the unit circle. Such metrics are called *co-axial*.

Metrics on the sphere for which equality holds in (5) are always co-axial.

For a co-axial monodromy, there is always an infinite set of transformations in $PSL(2)$ which commute with all monodromy transformations. Composing transformations of this set with the developing map, we obtain an infinite family of metrics with the same angles in the same conformal class. This family is actually one-dimensional in the case of non-trivial monodromy, and three-dimensional in the case of trivial monodromy.

Let us call two metrics with developing maps f_1, f_2 *equivalent* if $f_1 = \phi \circ f_2$ for some $\phi \in PSL(2)$.

Conjecture 1. *For any given angles and any given point in* $\mathrm{Mod}_{g,n}$ *the number of equivalence classes of metrics in the preimage of this point under the forgetful map is finite.*

In general, $\mathrm{Sph}_{g,n}(\boldsymbol{\alpha})$ has no natural complex analytic structure such that the forgetful map is analytic.

One case is known when such structure exists, and in fact $\mathrm{Sph}_{g,n}(\boldsymbol{\alpha})$ and F are algebraic in this case.

Example 1, continued. For metric on the sphere with all α_j integers, the monodromy is trivial, and the developing map is a rational function, and the singularities are critical points of this rational function of multiplicities $m_j = \alpha_j - 1$. So the problem in this case becomes: *how many equivalence classes of rational functions exist with prescribed critical points (of given multiplicity)?*. The answer is called the Kostka number $K(m_1, \ldots, m_n)$, there is no simple analytic expression for it but it can be simply described as the number of ways to fill a rectangular table of size $2 \times (2d-2)$ with numbers $1, \ldots, n$ so that the number k occurs m_k times, and so that the entries are strictly increasing in columns and non-decreasing in rows.

When $m_k = 1$ for all $k \in \{1, 2d-2\}$, we have an explicit expression

$$K(1, \ldots, 1) = \frac{(2d-2)!}{d(d-1)!},$$

the d-th Catalan number.

Theorem 6. (I. Scherbak) *For given integers* $m_1, \ldots, m_n$ *such that*

$$\sum_{k=1}^{n} m_k = 2d - 2$$

there is at most $K(m_1, \ldots, m_n)$ *equivalence classes of rational functions with critical points of order* m_k *at the prescribed points* a_k*. For generic points* a_k*, there are precisely* $K(m_1, \ldots, m_n)$ *classes.*

This has been generalized to the case of the sphere with any number of singularities, all but 3 of them with integer angles.

Theorem 7. (Eremenko and Tarasov [6]) *If $\alpha_1, \alpha_2, \alpha_3$ are not integers, while $\alpha_4, \ldots, \alpha_n$ are integers, then the necessary and sufficient conditions of existence on a non-coaxial metric on the sphere with these angles are (4) and (5) with strict inequality. The number of these metrics with given singularities is at least* 1 *and at most $\alpha_1 \ldots, \alpha_n$, and it is equal to $\alpha_4 \ldots \alpha_n$ for generic position of singularities.*

This theorem contains all cases when the forgetful map is known to be complex analytic with respect to an appropriate complex structure on $\mathrm{Sph}_{g,n}(\boldsymbol{\alpha})$.

Example 3. Consider the angles $2\pi(1/2, 1/2, 1/2, m)$ on the sphere, where $m \geq 2$ is an integer. Theorem 7 implies that there are m metrics with these angles for generic location of singularities. These metrics can be lifted on a torus via the standard 2-to-1 ramified covering, and the resulting metric on the torus will have only one singularity with angle $4\pi m$. One can show that all equivalence classes of metrics of positive curvature with one singularity on a torus can be obtained by such liftings. According to Theorem 7, there are m such classes for generic singularities. This can be compared with Theorem 5, which gives m as the degree of the forgetful map in this case.

There is a complete description of metrics on the sphere with 3 singularities: each such metric is obtained by gluing a spherical triangle with its reflection, and all spherical triangles are described by the following

Theorem 8. (Klein and Eremenko [3]) **1.** *If none of the $\alpha_1, \alpha_2, \alpha_3$ is an integer then the necessary and sufficient condition of existence of a spherical triangle with angles $\pi\alpha_j$ is*

$$\cos^2 \pi\alpha_1 + \cos^2 \pi\alpha_2 + \cos^2 \pi\alpha_3 + 2 \cos \pi\alpha_1 \cos \pi\alpha_2 \cos \pi\alpha_3 < 1.$$

A triangle with such angles is unique.

2. *If α_1 is an integer while α_2, α_3 are not then a spherical triangle with angles $\pi\alpha_j$ exists if and only if either $\alpha_2 + \alpha_3$ or $|\alpha_2 - \alpha_3|$ is an integer m of opposite parity to α_2, and $m \leq \alpha_1 - 1$. For any such angles a* 1*-parametric family of triangles exists.*

3. *If two of the α_j are integers then all three are integers, and the necessary and sufficient condition of existence of a spherical triangle with angles $\pi\alpha_j$ is that $\alpha_1+\alpha_2+\alpha_3$ is odd, and*

$$\max\{\alpha_1,\alpha_2,\alpha_3\} \leq (\alpha_1+\alpha_2+\alpha_2-1)/2.$$

There is a two-parametric family of triangles with given angles in this case.

There are only few cases when all questions stated in the beginning were completely answered: besides those already listed the case of torus with one singularity with angle 6π is completely understood. In this last case we have

Theorem 9. (Lin and Wang) *Depending on the conformal modulus of the torus, there is one or none classes of conformal metrics of curvature* 1 *with one conic singularity with angle 6π. The curve in the τ plane separating these two possibilities is described by the equation*

$$\mathrm{Im}\frac{\eta_1+\omega_1 e_j}{\eta_2+\omega_2 e_j} = 0, \quad j \in \{1,2,3\}. \tag{10}$$

In particular, there is such a metric on a hexagonal torus, and no metric on the square one.

This was first proved in [7] with the PDE methods. Then a much shorter proof was proposed in [1].

We describe the idea of this proof. Equation (9) in this case is the Lamé equation

$$w'' - (2\wp(z)+\lambda)\, w = 0.$$

This equation can be explicitly solved, which gives

$$f(z) = e^{2z\zeta(a)}\frac{\sigma(z-a)}{\sigma(z+a)}, \quad \text{where} \quad \lambda = \wp(a).$$

are pure imaginary.A simple computation shows that the monodromy is unitary iff one complex linear equation

$$Aa + B\overline{a} + \zeta(a) = 0, \tag{11}$$

holds, where

$$A = \frac{\pi}{4\omega_1^2\mathrm{Im}\tau} - \frac{\eta_1}{\omega_1}, \quad \text{and} \quad B = -\frac{\pi}{4|\omega_1|^2\mathrm{Im}\tau}.$$

These are unique constants which make the LHS of (11) periodic.

Equation (11) has another interpretation. The *Green function* G of the torus is defined as the real solution of

$$\Delta G = -\delta + 1/|T|,$$

where δ is the delta-function, and $|T|$ is the area of the torus with respect to the flat metric. The question is how many critical points can G have? The answer is 3 or 5, depending on the conformal modulus of the torus, and it is amazing that this answer was obtained only in 2012!

One can show that the gradient of G is exactly the LHS of (11).

Notice that the LHS of (11) is an odd function, and every odd periodic function is zero at the half-periods, so (11) always has 3 solutions which are called trivial.

Equation (11) can be re-written as the fixed point equation of the anti-holomorphic map

$$a \mapsto -\overline{(\zeta(a) + Aa)/B} =: g(a).$$

This is where (anti)-holomorphic dynamics comes into play. Function g is anti-holomorphic, so its second iterate is holomorphic, and Fatou's theorem from holomorphic dynamic applies. Since the derivative $\wp - A$ has two zeros on the torus, Fatou's theorem implies that there are at most two *attracting* fixed points.

On the other hand, consider the map $\phi(z) = z - g(z) : T \to \overline{\mathbf{C}}$. It has a degree and looking at preimage of ∞ we conclude that the degree is -1. This map preserves or reverses orientation, depending on the sign of its Jacobian. It is easy to see that $J = 1 - |\overline{\partial} g|^2$, so the fixed points in the region where orientation is preserved are attracting. If N^+ and N^- are the numbers of orientation preserving and reversing zeros of ϕ, then the degree formula gives $N^+ - N^- = -1$, while Fatou theorem implies that $N^+ \leq 2$. This shows that the total number of zeros is

$$N = N^+ + N^- = 2N^+ + 1 \leq 5,$$

as advertised.

Trivial solutions do not give any metrics, while non-trivial are $a, -a$ (since the function is odd) and they give one metric.

The separating line (10) is obtained from the condition that g has a neutral fixed point.

To conclude, we briefly mention the original problem of classification of polygons. By gluing a polygon to its reflection we obtain a metric with a special property: its conic singularities lie on a circle of the Riemann sphere, and the metric itself is symmetric with respect to the reflection in this circle. When all angles are integers, we have a remarkable fact: each class of metrics whose singularities lie on a circle contains exactly one symmetric representative [5]. This fact does not generalize to the metrics with non-integer angles.

References

[1] W. Bergweiler and A. Eremenko, Green's function and anti-holomorphic dynamics on a torus, Proc. Amer. Math. Soc., 144 (2016) 7, 2911–2922.

[2] C.-C. Chen and C.-S. Lin, Mean field equation of Liouville type with singular data: topological degree, Comm. Pure Appl. Math. 68 (2015), no. 6, 887–947.

[3] A. Eremenko, Metrics of positive curvature with conic singularities on the sphere, Proc. AMS, 132 (2004), 11, 3349–3355.

[4] A. Eremenko, Co-axial monodromy, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 20 (2020), no. 2, 619–634.

[5] A. Eremenko and A. Gabrielov, An elementary proof of the B. and M. Shapiro conjecture for rational functions, Notions of positivity and the geometry of polynomials, 167–178, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2011.

[6] A. Eremenko and V. Tarasov, Fuchsian equations with three non-apparent singularities, SIGMA (Symmetry Integrability Geom. Methods Appl.) 14 (2018), Paper No. 058, 12 pp.

[7] C.-S. Lin and Wang, Elliptic functions, Green functions and the mean field equations on tori, Ann of Math., 172 (2010) 2, 911–954.

[8] Feng Luo and Gang Tian, Liouville equation and spherical convex polytopes, Proc. Amer. Math. Soc. 116 (1992), no. 4, 1119–1129.

[9] G. Mondello and D. Panov, Spherical metrics with conical singularities on a 2-sphere: angle constraints, Int. Math. Res. Not. IMRN 2016, no. 16, 4937–4995.

[10] G. Mondello and D. Panov, Spherical surfaces with conical points: systole inequality and moduli spaces with many connected components, Geom. Funct. Anal. 29 (2019), no. 4, 1110–1193.

[11] E. Picard, Quelques applications analytiques de la théorie des courbes et des surfaces algébriques. Lecons redigees par Jean Dieudonne, Paris: Gauthier-Villars 224 p. (1931).