# Lectures on Nevanlinna theory

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Value distribution of a rational function $f$ is controlled by its degree $d$, which is the number of preimages of a generic point. If we denote by $n(a)$ the number of solutions of the equation $f(z) = a$, counting multiplicity, in the complex plane $\mathbf{C}$, then $n(a) \leq d$ for all $a \in \overline{\mathbf{C}}$ with equality for all $a$ with one exception, namely $a = f(\infty)$. The number of critical points of $f$ in $\mathbf{C}$, counting multiplicity, is at most $2d - 2$.

Nevanlinna theory generalizes these facts to transcendental functions $f : \mathbf{C} \mapsto \overline{\mathbf{C}}$. The main tool is the characteristic function $T_f(r)$ which replaces the degree in the case when $f$ is transcendental.

## 1. Jensen's formula

Let us denote by $n(r, a) = n_f(r, a)$ the number of solutions of the equation $f(z) = a$ in the disk $\{z : |z| \leq r\}$, counting multiplicity. Here $a \in \overline{\mathbf{C}}$. By the Argument Principle and the Cauchy–Riemann equations we have for $a \neq \infty$:

$$n(r, a) - n(r, \infty) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f'}{f - a} dz = \frac{r}{2\pi} \frac{d}{dr} \int_{-\pi}^{\pi} \log |f(re^{i\theta}) - a| \, d\theta. \quad (1)$$

We divide by $r$ and integrate with respect to $r$, assuming for a moment that $f(0) \neq a, \infty$ and using the notation[1]

$$N(r, a) = \int_0^r \frac{n(t, a)}{t} dt$$

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[1]If $f(0) = a$ this has to be regularized in the following way:

$$N(r, a) = \int_0^r \{n(t, a) - n(0, a)\} t^{-1} dt + n(0, a) \log r.$$

to obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})-a|\,d\theta=\log|f(0)-a|+N(r,a)-N(r,\infty). \tag{2}$$

This is the Jensen formula.[2]

Let us sketch another proof of it. It is enough to prove the formula with $a=0$. First we verify the formula for functions which have no zeros and no poles in $\{z:|z|\le r\}$. In this case, Jensen's formula is just the average property of harmonic functions. To derive the general case, consider the factors

$$g_c(z)=\frac{r(z-c)}{r^2-\bar{c}z},\quad |c|<r.$$

This function has a single simple zero at $z=c$ and satisfies $|g(z)|=1$ for $|z|=r$. Let $c_1,\ldots,c_n$ and $b_1,\ldots,b_m$ be the zeros and poles of $f$ in $\{z:|z|<r\}$, repeated according to their multiplicities. Then

$$g(z)=f(z)\frac{\prod_{k=1}^m g_{b_k}(z)}{\prod_{k=1}^n g_{c_k}(z)}$$

is free of zeros and poles. Applying the Jensen formula to $g$ we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|\,d\theta=\log|f(0)|+\sum_{k=1}^{n}\log\frac{r}{|c_k|}-\sum_{k=1}^{m}\log\frac{r}{|b_k|}.$$

Now we integrate by parts:

$$\begin{aligned}\int_0^r\frac{n(t,0)}{t}dt &= \int_0^r n(t,0)\,d\log t\\ &= n(r,0)\log r-\int_0^r\log t\,dn(t,0)\\ &= \sum_{k=1}^{n}\log\frac{r}{|c_k|}.\end{aligned}$$

Jensen's formula is closely related to Green's formula which we write in the form

$$\int\!\!\int_{|z|<r}\Delta u\,dm=\int_{|z|=r}\frac{\partial u}{\partial n}ds. \tag{3}$$

Here $dm$ is the area element, $ds$ is the length element, and $\partial/\partial n=\partial/\partial r$ is the differentiation in the direction of the outer normal. If $u=\log|f|$, the Laplacian $\Delta u$ has to be interpreted as a distribution:

$$\Delta\log|z|=2\pi\delta,$$

[2]If $f(0)=a$ and $f(z)-a=cz^m+\ldots, c\neq 0$, the term $\log|f(0)-a|$ has to be replaced by $\log|c|$, and the definition of $N(r,a)$ in the footnote on the previous page has to be used.

where $\delta$ is the unit charge at 0. Then (3) coincides with (1). Dividing by $r$ and integrating (3) we obtain

$$\int_{-\pi}^{\pi} u(re^{i\theta})d\theta = \int_0^r \frac{dt}{t} \int\int_{|z|\leq t} \Delta u dm + u(0). \tag{4}$$

This makes sense when $u$ is a difference of two subharmonic functions and $u(0) \neq 0, \infty$.

**2. First main theorem of Nevanlinna**

Using the notation $x^+ = \max\{x, 0\}$, we define

$$m_f(r, \infty) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta, \tag{5}$$

and

$$m_f(r, a) = m_{(f-a)^{-1}}(r, \infty) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} d\theta,$$

Using the relation

$$\log^+(|x \pm y|) \leq \log^+ |x| + \log^+ |y| + \log 2 \tag{6}$$

one obtains

$$m_f(r, \infty) = m_{f-a}(r, \infty) + O(1),$$

so (2) can be rewritten as

$$m(r, \infty) + N(r, \infty) = m(r, a) + N(r, a) + O(1), \quad r \to \infty. \tag{7}$$

This is the first main theorem. It justifies the definition of the Nevanlinna characteristic:

$$T_f(r) := m_f(r, \infty) + N_f(r, \infty).$$

First main theorem says that the sum of two non-negative terms $N(r, a) + m(r, a)$ is roughly independent of $a$. The first term counts $a$-points, and the second measures the average proximity of $f(z)$ to $a$ on the circle $|z| = r$. $N(r, a)$ and $m(r, a)$ are called the counting function and the proximity function, respectively.

**Exercise 1.** For every non-constant $f$, $T_f(r) \to \infty$ as $r \to \infty$. If $f$ is a rational function of degree $d$, then

$$T_f(r) = d \log r + O(1). \tag{8}$$

If $f$ is transcendental, then

$$T_f(r)/\log r \to \infty.$$

**Exercise 2*.** (Goldberg) For $f(z) = e^z$, compute $T(r)$ and $N(r,1)$, and estimate $m(r,1)$ as accurately as you can. What does the First main theorem give for this case?

Algebraic properties of the characteristic are similar to those of the degree of a rational function. Using (6) and

$$\log^+ |xy| \leq \log^+ |x| + \log^+ |y|,$$

and counting the poles, we obtain

$$\begin{array}{rc} a) & T_{f^n} = nT_f, \\ b) & T_{fg} \leq T_f + T_g, \\ c) & T_{f+g} \leq T_f + T_g + O(1), \\ d) & T_{1/f} = T_f + O(1). \end{array}$$

The last property is Jensen's formula.

**Exercise 3***.** Let $K$ be the field of rational functions, and $T : K \to \mathbf{R}^+$ a function which satisfies a)-d) without the $O(1)$ terms, and $T_{\text{const}} = 0$. Prove that such $T$ is proportional to the degree.

**Exercise 4***.** (Mokhonko) Let $R(w,z)$ be a rational function of degree $d$ with respect to $w$, whose coefficients are meromorphic functions satisfying $T(r) = O(V(r))$, for some positive increasing function $V$. Let $f$ be a meromorphic function. Set $g(z) = R(f(z), z)$ and prove that

$$T_g(r) = dT_f(r) + O(V(r)).$$

Mokhonko's proof is purely algebraic. It uses only a)-d) above, but does not use the definition of $T$ or any properties of meromorphic functions. It extends to arbitrary fields with a function $T$ satisfying a)-d).

## 3. First main theorem in the form of Ahlfors and Shimizu

We explain the version of the First Main Theorem which was found by Shimizu and Ahlfors independently of each other.

Let $d\rho$ be a probability measure in $\mathbf{C}$. We integrate (2) with respect to $a \in \mathbf{C}$ against $d\rho$ and obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U(f(re^{i\theta}))\,d\theta = U(f(0)) + \int\int_{\mathbf{C}} N(r,a)\,d\rho(a) - N(r,\infty), \qquad (9)$$

where

$$U(w) = \int\int_{\mathbf{C}} \log|w-a|\,d\rho(a). \qquad (10)$$

If we choose $d\rho$ to be the normalized spherical area element that is

$$d\rho(w) = \frac{1}{\pi(1+|w|^2)^2}dudv, \quad w = u+iv,$$

then changing the order of integration shows

$$\mathring{T}(r) := \int\int_{\mathbf{C}} N(r,a)\,d\rho(a) = \int_0^r \frac{A(t)\,dt}{t}, \qquad (11)$$

where

$$A(r) = \frac{1}{\pi}\int_{|z|\leq r} \frac{|f'|^2}{(1+|f|^2)^2}dm, \quad \text{where } dm \text{ is the Euclidean area element in } \mathbf{C}.$$

The geometric interpretation of $A(r)$ is the area of the disk $|z| \leq r$ with respect to the pullback of the spherical metric, or in other words, the *average covering number* of the Riemann sphere by the restriction of $f$ to the disk $|z| \leq r$. The function $\mathring{T}(r)$ defined in (11) is called the *Ahlfors–Shimizu characteristic* of $f$. We will see in a moment that

$$\mathring{T}(r) = T(r) + O(1). \qquad (12)$$

Now the integrals in (10) can be evaluated:

$$\begin{aligned} U(w) &= \frac{1}{\pi}\int\int_{\mathbf{C}} \frac{\log|w-a|}{(1+|a|^2)^2}dm(a) \\ &= \log\sqrt{1+|w|^2} = \log([w,\infty])^{-1}, \end{aligned}$$

where $[,]$ stands for the *chordal* distance on the Riemann sphere.[3] Thus the first term in (9) is

$$\mathring{m}(r,\infty) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\frac{1}{[f(re^{i\theta}),\infty]}d\theta.$$

[3] The first step in this evaluation is $\int_{-\pi}^{\pi}\log|w-e^{i\theta}|d\theta = 2\pi\log^+|w|$, which follows from Jensen's formula. In our normalization, the diameter of the Riemann sphere is 1.

and in general we can define

$$\mathring{m}(r,a) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \frac{1}{[f(re^{i\theta}),a]}d\theta, \quad a \in \overline{\mathbf{C}}.$$

This is called the *proximity function in the sense of Ahlfors–Shimizu* It becomes large when $f$ is close to $a$ in the average on the circle $|z| = r$. It is important that the proximity function is non-negative. With these notations (9) can be rewritten as

$$N(r,\infty) + \mathring{m}(r,\infty) = \mathring{T}(r) + \mathring{m}(0,\infty).$$

It is easy to see that $\mathring{m}(r,\infty) = m(r,\infty) + O(1)$, so (12) follows. Now we notice that $T(r)$ does not change if we replace $f$ by $L \circ f$ where $L$ is a rotation of the sphere, which is a conformal automorphism which preserves spherical distance. Thus we obtain

$$N(r,a) + \mathring{m}(r,a) = \mathring{T}(r) + \mathring{m}(0,a) = \mathring{T}(r) + O(1),\ r \to \infty,\ a \in \overline{\mathbf{C}}. \quad (13)$$

This is the First Main Theorem (FMT) of Nevanlinna in the form of Ahlfors–Shimizu. It implies by the way that

$$N(r,a) \leq \mathring{T}(r) + O(1),\ r \to \infty, \text{ for every } a \in \overline{\mathbf{C}}, \quad (14)$$

because the proximity function is non-negative. When compared with (11) this shows that the points $a$ for which $N(r,a)$ is substantially less then $\mathring{T}(r)$ should be exceptional.

We have seen that the difference between $T(r)$ and $\mathring{T}(r)$ is insignificant, and they can be replaced by each other in most statements.

## 4. Gauss–Bonnet Formula

Here we give a differential-geometric interpretation of Jensen's formula. It explains what is going on in the next section, though the next section can be read independently of this one.

A (Riemannian) metric is an assignment of a positive definite quadratic form to each tangent space. These quadratic forms can be used to measure angles and lengths of curves. Two metrics are called conformally equivalent if they are proportional. (The positive coefficient of proportionality depends on the point). We will call a metric simply *conformal* if it is conformally equivalent to the standard metric on the plane. Thus a conformal metric is locally described as

$$p^2(z)|dz|^2,$$

where $p$ is a positive smooth function. It gives the length of a curve $\gamma$

$$\int_\gamma p(z)\,|dz|$$

and the area of a set $E$

$$\int\int_E p^2(z)\,dxdy.$$

The (Gaussian) curvature of a metric is defined by

$$\kappa = -\frac{\Delta \log p}{p^2}.$$

This is a function of a point. The integral curvature of a set $E$ is

$$-\int_E \Delta \log p\, dxdy.$$

A. D. Alexandrov and his co-authors developed a theory of surfaces with minimal smoothness assumptions on the metric. In this theory, $p$ can be any non-negative function for which $\Delta \log p$ is a (signed) Borel measure, and which is integrable on every line segment. In other words,

$$p = e^u,$$

where $u$ is a difference of two subharmonic functions. In this setting, Green's formula (3) becomes the Gauss–Bonnet formula for the disc $|z| \leq r$. It relates the integral curvature of the disc and the geodesic curvature of its circumference.

**Exercise 5.** (Gauss–Bonnet theorem for the sphere). Consider a conformal metric on the sphere $\overline{\mathbf{C}}$, and prove that the integral curvature of the whole sphere is $4\pi$.

**Exercise 6.** Check that the metric

$$\frac{2|dz|}{1+|z|^2}$$

has curvature 1, and the metric

$$\frac{2|dz|}{1-|z^2|}, \quad |z| < 1$$

has curvature $-1$.

We only need a special case of Alexandrov's surfaces, surfaces with conic singularities. The metric on such surfaces is smooth and has a constant curvature, except the isolated singularities. Each isolated singularity has a neighborhood isometric to a neighborhood of the vertex of a cone.

The simplest example is a convex polyhedron in $\mathbf{R}^3$ with the intrinsic metric induced from $\mathbf{R}^3$. The metric is of zero curvature everywhere except the vertices. Each vertex contributes an atom to the integral curvature, of mass $2\pi - \alpha$, where $\alpha$ is a total angle around the vertex.

In our situation, let $p(w)|dw|$ be a smooth metric in a neighborhood of 0, and let $w = f(z)$ be an analytic function with zero of order $n$ at 0. Then the pull back metric has the form

$$p(f(z))|f'(z)||dz|$$

and the integral curvature $-\Delta \log p$ has an atom at 0 of mass $2\pi(1-n)$, and the total angle around 0 in the pull-back metric is $2\pi n$.

Using these considerations, let us prove a weak version of the Second main theorem. Let $n_1(r)$ be the counting function of critical points of a meromorphic function $f$, and

$$N_1(r) = \int_0^r \frac{n_1(t)}{t} dt, \tag{15}$$

as usual.

We will prove that

$$N_1(r) \leq (2 + o(1))T(r), \tag{16}$$

when $r \to \infty$, $r \notin E$, where $E$ is an exceptional set of finite length.

*Proof.* The pull-back of the spherical metric is

$$p(z)|dz| = \frac{2|f'||dz|}{1 + |f|^2}$$

has curvature 1 away from the critical points and each critical point contributes $-2\pi m$, where $m$ is the multiplicity of the critical point. So we have

$$\int\int_{|z|<t} \Delta \log p \, dxdy = -\int\int_{|z|<t} p^2 \, dxdy + 2\pi n_1(t),$$

in the sense of distributions. Dividing by $t$ and integrating from 0 to $r$, we obtain in the RHS

$$-4\pi T(r) + 2\pi N_1(r),$$

and it remains to show that the LHS is at most $o(T(r))$. This estimate is crucial for the whole business.

According to the Gauss–Bonnet (Jensen) formula (3), this LHS is a constant times

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{|f'|}{1 + |f|^2} d\theta.$$

First we apply the inequality between arithmetic and geometric averages, and obtain that the last expression is at most

$$\log \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f'|}{(1 + |f|^2)} d\theta =: \log \frac{1}{2} \lambda(r).$$

Now we notice that

$$\lambda(r) = (rT')'/r,$$

which follows from (11), and use the following lemma

**Lemma.** *If $g$ is an increasing function on $[0, \infty)$, tending to $+\infty$, and $\epsilon > 0$, then $g'(x) \leq g^{1+\epsilon}(x)$ for all $x \notin E$, where $E$ is a set of finite measure.*

*Proof.* Let $E$ be the set where $g'(x) \geq g^{1+\epsilon}(x)$, then

$$\int_E dz \leq \int_E \frac{g'(x)}{g^{1+\epsilon}}(x)dx = \int \frac{dy}{y^{1+\epsilon}} < \infty.$$

Applying this lemma twice we conclude that $\lambda(r) < rT(r)^{1+\epsilon}$, $r \notin E$, which proves (16) with the error term $\log rT(r)$.

## 5. The Second Main Theorem of the value distribution theory

To formulate the main result of the value distribution theory we recall that $n_1(r) = n_{1,f}(r)$ denotes the number of critical points of a meromorphic function $f$ in the disk $|z| \leq r$, counting multiplicity. It is easy to check that

$$n_{1,f}(r) = n_{f'}(r, 0) + 2n_f(r, \infty) - n_{f'}(r, \infty). \tag{17}$$

Now we apply the averaging as above:

$$N_1(r) = N_{1,f}(r) := \int_0^r \frac{n_1(t)\, dt}{t}.$$

If 0 is a critical point, the same regularization as before has to be made. The Second Main Theorem (SMT) says that *for every finite set* $\{a_1, \ldots, a_q\} \subset \overline{\mathbf{C}}$ *we have*

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r) \leq 2T(r) + S(r), \tag{18}$$

*where* $S(r) = S_f(r)$ *is a small error term,* $S_f(r) = O(\log(rT(r)))$ *when* $r \to \infty, r \notin E$, *where* $E \subset [0, \infty)$ *is a set of finite measure*[4]. The SMT may be regarded as a very precise way of saying that the term $m(r, a)$ in the FMT (13) is relatively small for most $a \in \overline{\mathbf{C}}$. It is instructive to rewrite (18) using (13) in the following form. Let $\overline{N}(r, a)$ be the averaged counting function of *distinct* solutions of $f(z) = a$, that is this time we don't count multiplicity. Then $\sum_j N(r, a_j) \leq \sum_j \overline{N}(r, a_j) + N_1(r)$ and we obtain

$$\sum_{j=1}^{q} \overline{N}(r, a_j) \geq (q - 2)T(r) + S(r). \tag{19}$$

Now Picard's theorem is an immediate consequence: if three values $a_1, a_2$ and $a_3$ are omitted by a meromorphic function $f$, then $N_f(r, a_j) \equiv 0,\ 1 \leq j \leq 3$, so the left hand side of (19) is zero and we obtain $T_f(r) = S_f(r)$, which implies that $f$ is constant. Similarly, if the three equations $f(z) = a_j$ have only finitely many solutions, we conclude that $f$ is rational. Here is a more refined

[4]In fact $S(r)$ has more precise estimate. Recently there was a substantial activity in the study of the best possible estimate of this error term. On the other hand Hayman's examples (2.8) show that in general the error term may not be $o(T(r))$ for all $r$, so an exceptional set $E$ is really required.

**Corollary from the SMT.** *Let $a_1, \ldots, a_5$ be five points on the Riemann sphere. Then at least one of the equations $f(z) = a_j$ has simple solutions.*

Indeed, if all five equations have only multiple solutions then $N_1(r, f) \geq (1/2) \sum_{j=1}^{5} N(r, a_j)$. When we combine this inequality with SMT (18) it implies $(5/2)T(r) \leq 2T(r) + S(r)$, so $f = \text{const}$.

**Exercise 7.** *a) Suppose that for several values of $a_j$, all solutions of $f(z) = a_j$ have multiplicities at least $m_j \geq 2$. If some $a_j$ is omitted, we can set $m_j = \infty$. Then*

$$\sum_j \left(1 - \frac{1}{m_j}\right) \leq 2.$$

*b) The last inequality has finitely namy solutions. For each of them, there is a meromorphic function satisfying the stated condition.*

For most "reasonable" functions, like Nevanlinna's functions described in Theorem (3.1), the SMT tends to be an asymptotic equality rather then inequality; the most general class of functions for which this is known consists of meromorphic functions whose critical and singular points lie over a finite set. This is due to Teichmüller.

*Ahlfors' proof of SMT.*

We consider the area element

$$d\rho = p^2(w) \frac{dxdy}{\pi(1 + |w|^2)^2},$$

where $p$ is given by

$$\log p(w) := \sum_{j=1}^{q} \log \frac{1}{[w, a_j]} - 2 \log \left( \sum_{j=1}^{q} \log \frac{1}{[w, a_j]} \right) + C, \qquad (20)$$

where $[,]$ is the chordal distance, and $C > 0$ is chosen so that

$$\int \int_{\mathbf{C}} d\rho = 1.$$

(The sole purpose of the second term in the definition of $p$ in (20) is to make this integral converge, without altering much the behavior near $a_j$ which is determined by the first term). We pull back this $d\rho$ via $f$ and write the change of the variable formula:

$$\int\int_{\mathbf{C}} n(r,a)d\rho(a) = \int_0^r \int_{-\pi}^{\pi} p^2(w)\frac{|w'|^2}{(1+|w|^2)^2} t\, d\theta dt, \quad w = f(te^{i\theta}). \tag{21}$$

Now we consider the derivative of the last double integral with respect to $r$, divided by $2\pi r$:

$$\lambda(r) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|w'|^2}{(1+|w|^2)^2} p^2(w)\, d\theta, \quad w = f(re^{i\theta}).$$

Using the integral form of the arithmetic-geometric means inequality[5] we obtain

$$\log \lambda(r) \geq \frac{1}{\pi}\int_{-\pi}^{\pi} \log p(w)\, d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi} \log(1+|w|^2) d\theta + \frac{1}{\pi}\int_{-\pi}^{\pi} \log|w'|\, d\theta. \tag{22}$$

The first integral in the right-hand side of (22) is approximately evaluated using (20); the second summand in (20) becomes irrelevant because of another log:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \log p(f(re^{i\theta})) d\theta = 2\sum_{j=1}^{q} \mathring{m}(r,a_j) + O(\log T(r)).$$

the second integral equals $4m(r,\infty)$ and the third one is evaluated using Jensen's formula (2). This gives

$$2\sum_{j=1}^{q} m(r,a_j) + 2\left\{N(r,0,f') - N(r,\infty,f') - 2m(r,\infty)\right\} \leq \log\lambda(r) + O\log T(r).$$

The expression inside the brackets is equal to $N_1(r) - 2T(r)$ (by definition of $N_1$ and the FMT (13) applied with $a=\infty$), so

$$\sum_{j=1}^{q} m(r,a_j) + N_1(r) - 2T(r) \leq \frac{1}{2}\log\lambda(r). \tag{23}$$

To estimate $\lambda$ we return to the left side of (21). Using (9) we integrate and obtain

$$\int_0^r \frac{dt}{t}\int_0^t \lambda(s) s ds = \int_{\overline{\mathbf{C}}} N(r,a)\, d\rho(a) \leq N(r,\infty) + \frac{1}{2\pi}\int_{-\pi}^{\pi} U(f(re^{i\theta})) d\theta.$$

[5] $\frac{1}{b-a}\int_a^b \log g(x)\, dx \leq \log\left\{\frac{1}{b-a}\int_a^b g(x)\, dx\right\}$

But $U(w)$ is a potential of a probability measure $d\rho$, so $U(w) \leq \log^+ |w| + O(1)$ and we obtain

$$\int_0^r \frac{dt}{t} \int_0^t \lambda(s) s ds \leq T(t) + O(1).$$

Now the argument is concluded with the application of the Lemma 1 of the previous section. Applying this lemma twice we conclude that $\log \lambda(r) = S(r)$ which proves the theorem.

**5. Functions in the unit disc**

For meromorphic functions in the unit disc, we only need to modify Lemma 1.

**Lemma 2.** *Let $g$ be an increasing function on $[0, 1)$, tending to $+\infty$, and $\epsilon > 0$. Then $g'(r) \leq g^{1+\epsilon}(r)/(1-r)$ for all $r \notin E$, where $E \subset [0, 1)$ is a set such that*

$$\int_E \frac{dr}{1-r} < \infty.$$

*Proof.* Let $E$ be the set where the opposite inequality holds: $g'(r) > g^{1+\epsilon}(r)/(1-r)$. Then

$$\int_E \frac{dr}{1-r} \leq \int_E \frac{g'(r)}{g^{1+\epsilon}}(r) dr = \int \frac{dy}{y^{1+\epsilon}} < \infty.$$

Thus we get the Second main theorem of the unit disc with the error term $S(r) = O\left(\log T(r) + \log \frac{1}{1-r}\right)$, $r \notin E$.

As a corollary, we obtain that a holomorphic function $f$ in the unit disc that omits 0 and 1 must satisfy

$$T_f(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta = O(\log(1-r)^{-1}).$$

It is instructive to compare this with Schottky's theorem which gives the best possible uniform bound

$$\log |f(re^{i\theta})| = O((1-r)^{-1}).$$

Thus by averaging $\log |f(re^{i\theta})|$ with respect to $\theta$ we gain a log in the upper estimate.

An example which shows that Schottky's bound is best possible is $f(z) = \exp\dfrac{1+z}{1-z}$. For this function, $T_f$ is bounded. The simplest example of a function omitting 0 and 1 and with $T(r) \sim \log(1-r)^{-1}$ is the modular function.

## 6. Lemma on the logarithmic derivative

In this section we outline Nevanlinna's original proof of the Second main theorem which is based on the Lemma on the logarithmic derivative. This lemma seems to be a more versatile tool than the Ahlfors technique in the previous section.

**Theorem.** (Lemma on the logarithmic derivative).

$$m_{f'/f}(r) = S_f(r).$$

We deduce the Second main theorem from this. We have

$$\begin{aligned} T_{f'}(r) &= N_{f'}(r,\infty) + m_{f'}(r,\infty) \\ &\leq 2N_f(r,\infty) + m_{f'f/f}(r,\infty) \\ &\leq 2N_f(r,\infty) + m_f(r,\infty) + m_{f'/f}(r,\infty) + O(1) \\ &\leq 2T_f(r) + S(r), \end{aligned}$$

so we obtained the result (16) at the end of section 4.

In proving the Second Main theorem, we may assume without loss of generality that all poles of $f$ are simple. This is achieved by a fractional-linear transformation. Fix $a_1, \ldots, a_q$ in $\mathbf{C}$, and consider the auxiliary function

$$g := \sum_{j=1}^{q} \frac{1}{f - a_j}.$$

If $\delta = \min|a_i - a_j|$ then the inequality $|f(z) - a_j| \leq \delta/2$ cannot hold for more than one $j$, so we conclude

$$\begin{aligned} \sum_{j=1}^{q} m_f(r,a_j) &\leq m_g(r,\infty) + O(1) = m_{f'g/f'}(r,\infty) + O(1) \\ &\leq m_{f'g}(r,\infty) + m_{1/f'}(r,\infty) + O(1) \\ &\leq T_{f'}(r) - N_{f'}(r,0) + S(r) \leq 2T_f(r) - N_1(r) + S(r). \end{aligned}$$

This completes the derivation of the SMT from the Lemma on the logarithmic derivative.

**Corollary.** $T(r, f') \leq 2T(r, f) + S(r, f)$.

## 7. H. Cartan's generalization to holomorphic curves

Cartan generalized the two main theorems of Nevanlinna theory to holomorphic maps $f : \mathbf{C} \to \mathbf{P}^n$ (holomorphic curves). Such maps can be represented in homogeneous coordinates as $f = (f_0, \ldots, f_n)$ where $f_j$ are entire functions without common zeros. These functions are defined up to a common factor which is an entire function without zeros. Cartan's theory deals with preimages of hyperplanes. A hyperplane $A$ in $\mathbf{P}^n$ is described by a linear homogeneous equation

$$a_0 w_0 + \ldots + a_n w_n = 0. \tag{24}$$

So $f$-preimages of this hyperplane are zeros of the linear combination $g_A = (f, A) = a_0 f_0 + \ldots + a_n f_n$.

Let $\|f\| = \sqrt{|f_0|^2 + \ldots + |f_n|^2}$ be the Euclidean norm and consider the subharmonic function $u = \log \|f\|$. We have by straightforward computation

$$\Delta u = 2\|f\|^{-4} \sum_{i<j} |f_i' f_j - f_i f_j'|^2 =: 2\|f'\|^2.$$

The last expression can be called the "Fubini–Study derivative".

Cartan defines the characteristic by the formula

$$T_f(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) d\theta - u(0) = \int_0^r \frac{A(t)}{t} dt, \tag{25}$$

where

$$A(t) = \frac{1}{2\pi} \int \int_{|z| \leq t} \Delta u \, dm = \frac{1}{\pi} \int \int_{|z| \leq t} \|f'\|^2 dm.$$

Equality in (25) holds by Jensen's formula (4).

**Exercise 8.** Verify that $\|f'\|$ and $T(r)$ are independent of the homogeneous representation and depend only on $f$.

**Exercise 9.** For $n = 1$, check that $\|f'\| = |f'|/(1 + |f|^2)$, where $f = f_0/f_1$ and $f_0$ and $f_1$ are holomorphic functions without common zeros. So (25) gives a new characterization of $\mathring{T}(r)$ for $n = 1$:

$$\mathring{T}(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{|f_0|^2 + |f_1|^2} (re^{i\theta}) d\theta.$$

For a hyperplane $A$ described by the equation (24) and such that $f(\mathbf{C}) \notin A$, we introduce the counting function

$$N(r, A) = N_{(f,A)}(r, 0), \quad (f, A) = a_0 f_0 + \ldots + a_n f_n$$

and the proximity function

$$m(r, A) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (u + \log \|A\| - \log |(f, A)|)(re^{i\theta}) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{\|f\| \|A\|}{|(f, A)|}(re^{i\theta}) d\theta,$$

where $\|A\| = |a_0|^2 + \ldots + |a_n|^2$. The expression under the logarithm in the last integral is reciprocal to

$$\frac{|(f, A)|}{\|f\| \|A\|},$$

which is the sine of the angle between the vector $f$ and the hyperplane $A$. This is a distance[6] in $\mathbf{P}^n$, so the proximity function has the same interpretation as in dimension 1: integral of the logarithm of the reciprocal distance.

The First main theorem of Cartan,

$$m(r, A) + N(r, A) = T(r) + O(1),$$

is now a direct consequence of the Jensen formula

$$N(r, A) = N_{(f,A)}(r, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |(f, A)(re^{i\theta})| d\theta + O(1).$$

To state the Second main theorem of Cartan, we define $n_1(r)$ as the number of zeros, counting multiplicity, of the Wronski determinant

$$W = W(f_0, \ldots, f_n) = \begin{vmatrix} f_0 & \ldots & f_n \\ f_0' & \ldots & f_n' \\ f_0^{(n)} & \ldots & f_n^{(n)} \end{vmatrix}.$$

Then $N_1$ is defined as in (15).

We say that hyperplanes $A_1, \ldots, A_q$ are *admissible* if $q > n$, and the intersection of any $(n + 1)$ of these hyperplanes is empty.

Now we can state the Second main theorem:

[6]Verify that it equals the chordal distance when $n = 1$.

*Let $f$ be a holomorphic curve in $\mathbf{P}^n$ whose image does not belong to any hyperplane. Let $A_1, \ldots, A_q$ be an admissible collection of hyperplanes. Then*

$$\sum_{j=1}^{q} m(r, A_j) + N_1(r) \leq (n+1)T(r) + S(r).$$

When $n = 1$ this is exactly the same as the SMT of Nevanlinna.

We will use the following properties of the Wronski determinants.

a) $W(f_0, \ldots, f_n) \not\equiv 0$ iff $f_0 \ldots, f_n$ are linearly independent.

b) If $(g_0, \ldots, g_n) = (f_0, \ldots, f_n)B$, where $B$ is a constant $(n+1) \times (n+1)$ matrix, then $W(g_0, \ldots, g_n) = \det B W(f_0, \ldots, f_n)$, and

c) $W(gf_0, \ldots, gf_n) = g^{n+1} W(f_0, \ldots, f_n)$, for every function $g$.

d) $L(f_0, \ldots, f_n) := W(f_0, \ldots, f_n)/(f_0 \ldots, f_n)$

$$= \left| \begin{array}{ccc} 1 & \ldots & 1 \\ f_0'/f_0 & \ldots & f_n'/f_n \\ f_0^{(n)}/f_0 & \ldots & f_n^{(n)}/f_n \end{array} \right|.$$

e) $L(gf_0, \ldots, gf_n) = L(f_0, \ldots, f_n)$. This follows from c).

**Lemma.** *For a linearly non-degenerate curve $f = (f_0, \ldots, f_n)$, we have*

$$m(r, L(f_0, \ldots, f_n)) = S_f(r).$$

Proof. Without loss of generality, $f_0 \not\equiv 0$. By property e), we have $L(f_0, \ldots, f_n) = L(1, f_1/f_0, \ldots, f_n/f_0)$, which is by property d) a polynomial in higher logarithmic derivatives of $f_j/f_0$. Applying the Lemma on the logarithmic derivative we estimate $m(r, L)$ as $O(\sum_j \log(rT_{f_j/f_0})$. But we also have $T_{f_i/f_j}(r) \leq T(r, f)$ in view of (25), and the statement of the lemma follows.

Now we can complete the proof of the Second main theorem.

Denote $u_j = \log |(f, A_j)|$. Let $I \subset \{1, \ldots, q\}$, $|I| = n+1$. Then

$$\max_{j \in I} u_j = u + O(1). \tag{26}$$

Let $W_I$ be the Wronskian of the functions $(f, A_j), j \in I$, and $w_I = \log |W_I|$. Then by the property b) of the Wronskians we have $w_I = w + O(1)$, where $w = \log |W(f_0, \ldots, f_n)|$. Now we have

$$\begin{aligned}\sum_{j=1}^{n} u_j &= (q-n-1)u + \min_I \sum_{j\in I} u_j + O(1)\\ &= (q-n-1)u + \min_I (\sum_{j\in I} u_j - w_I) + w + O(1)\\ &\geq (q-n-1)u + w - \max_I (w_I - \sum_{j\in I} u_j)^+ + O(1)\\ &\geq (q-n-1)u + w - \sum_{I\subset\{1,\ldots,q\}} (w_I - \sum_{j\in I} u_j)^+ + O(1).\end{aligned}$$

Now we integrate this from $-\pi$ to $\pi$ with respect to $\theta$, use Jensen's formula and the Lemma:

$$\sum_{j=1}^{q} N(r, A_j) \geq (q-n-1)T(r) + N_1(r) - S(r),$$

which completes the proof.

Cartan's SMT implies the following generalization of Picard's theorem: If a holomorphic curve $f \to \mathbf{P}^n$ omits $n+2$ admissible hyperplanes, then the image of this curve is contained in a hyperplane.

One can deduce from this that a holomorphic curve omitting $2n+1$ admissible hyperplanes must be constant.

Another useful corollary is

**Borel's Theorem**. *Let $g_1, \ldots, g_p$ be entire functions, and*

$$e^{g_1} + e^{g_2} + \ldots + e^{g_p} = 0.$$

*Then the $\{g_j\}$ can be partitioned into disjoint groups such that for $g_i$ and $g_j$ in one group, the difference $g_j - g_i$ is constant, and the sum of $e_j^g$ over each group is* 0.

Remarks on the bibliography. Hayman's book remains the best introduction to one-dimensional theory. Cartan's theory is available in English in Lang's book. Goldberg and Ostrovski is a very comprehensive treatment; the English translation is equipped with a survey of results obtained in one-dimensional theory up to 2010. The contents of Nevanlinna's book of 1929 is not completely covered by his later book, in particular it contains an interesting discussion of (pre-Cartan) theory of holomorphic curves, as well as many other applications of the theory. The contents of section 5 of this survey is not contained in any English book that I know, so I refer to the original article of Ahlfors (1932). Ahlfors's method of proving the SMT can be also extended to holomorphic curves (Ahlfors, 1939); this proof is much more complicated than the proof of Cartan but it gives somewhat more general result. Cartan's thesis (1928) is about holomorphic curves in the unit disc. The last chapter of Lang's book has a detailed exposition of this work in English.

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