Holomorphic curves with few inflection points

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Consider a vector of linearly independent entire functions $f = (f_0, \ldots, f_n)$, whose coordinates have no common zeros. The *characteristic* is defined by

$$T(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \|f(re^{i\theta})\| \, d\theta - \log \|f(0)\|,$$

and the *lower order* is

$$\lambda = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}. \tag{1}$$

We also define

$$N_1(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |W(re^{i\theta})| d\theta,$$

where $W = W(f_0, \ldots, f_n)$ is the Wronskian determinant. N_1 is called the 'ramification term' in the Second Main Theorem of Cartan. Actually this is the averaged counting function of inflection points of the holomorphic curve $(f_0, \ldots, f_n)$ in projective space.

Conjecture. Suppose that $\lambda < \infty$ and $N_1(r) = o(T(r))$. Then λ is rational and the limit in (1) exists.

This is known to be the case when $n = 1$ [1].

Here is the motivation for the conjecture (the argument is due to V. P. Petrenko). Suppose that $\lambda < \infty$, and $N_1(r) \equiv 0$. This means that the Wronskian has no zeros, so $f_0, \ldots, f_n$ make a fundamental system of solutions to a differential equation

$$w^{(n+1)} + P_1 w^{(n)} + \ldots + P_0 w = 0,$$

where P_j are entire functions. By a theorem of M. Frei [2], our condition that $\lambda < \infty$ implies that P_j are polynomials. In this case, the conclusion of the conjecture is well known.

References

[1] A. Eremenko, Meromorphic functions with small ramification, Indiana Univ. Math. J., 42 (1993), 1193-1218.

[2] M. Frei, Sur l'ordre des solutions entières d'une équation diffèrentielle linéaire, C. r., 236 (1953), 38-40.