THE WRONSKI MAP, SCHUBERT CALCULUS AND POLE PLACEMENT

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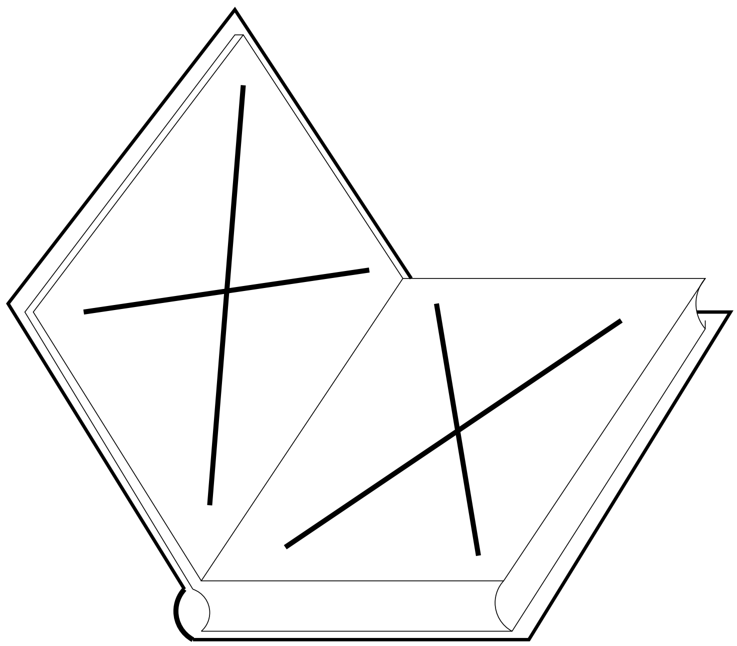
1. Given 4 generic affine lines in $\mathbf{C}^3$ or $\mathbf{R}^3$, how many lines intersect all of them? The answer over $\mathbf{C}$ is 2, over $\mathbf{R}$ either 2 or 0.



H. Schubert, Kalkül der abzählenden Geometrie

1879

2. Given mp linear p-subspaces in $\mathbf{C}^{m+p}$, how many linear m-spaces intersect all of them? (Previous problem corresponds to $m = p = 2$.)

Answer (Schubert, 1886):

$$N(m,p) = \frac{1!2!\ldots(p-1)!(mp)!}{m!(m+1)!\ldots(m+p-1)!}.$$

= the number of standard Young tableaux (SYT) of size $p \times m$.

Duality: $N(m,p) = N(p,m)$

$N(m,2) = \frac{1}{m+1}\binom{2m}{m}$ is the m-th Catalan number.

3. Let $F(z) = (1 : z : \ldots : z^d), \quad d = m + p - 1,$ be the rational normal curve in $\mathbf{C}^{m+p}$. Suppose that the given subspaces are osculating F at some points z_j. This means that the j-th subspace is spanned by the (row)-vectors $F(z_j), F'(z_j), \ldots, F^{(p-1)}(z_j)$.

MTV Theorem. (Mukhin, Tarasov, Varchenko; former B. & M. Shapiro conjecture). *Given mp p-subspaces osculating $F(z)$ at mp distinct real points, all m-subspaces intersecting these given ones are real.*

Proof is based on Bethe Ansatz for the Gaudin quantum integrable model. The m-subspaces are associated with common eigenvectors of commuting symmetric operators (Gaudin Hamiltonians), hence they are all real.

When $p = 2$ or $m = 2$ a proof based on different ideas was earlier given by Eremenko and Gabrielov.

Connection with differential equations. Let U be a vector space of polynomials, $\{f_1, \ldots, f_p\}$ a basis in U. Consider the differential equation with respect to w:

$$\begin{vmatrix} w & f_1 & \ldots & f_p \\ w' & f_1' & \ldots & f_p' \\ \ldots & \ldots & \ldots & \ldots \\ w^{(p)} & f_1^{(p)} & \ldots & f_p^{(p)} \end{vmatrix} = 0.$$

The space of solutions coincides with U. This equation is of the form

$$a_p w^{(p)} + \ldots + a_0 w = 0, \qquad (1)$$

where a_j are polynomials, $a_p = W(f_1, \ldots, f_p)$, the Wronski determinant. U has a real basis iff all a_j are real, up to a common constant factor. So Theorem MTV is equivalent to:

If in a differential equation (1), all coefficients are polynomials, and a_p is real with all zeros real, and all solutions of this equation are polynomials, then all a_j are real.

4. Let $E(z)$ be the $p \times (m+p)$ matrix with the rows $F^{(k)}(z)$, $k = 0, \ldots, p-1$. Let L be a constant $m \times (m+p)$ matrix whose rows span the unknown m-subspaces. Then our Schubert problem is equivalent to the following:
Find L from the conditions

$$\det \begin{pmatrix} E(z_j) \\ L \end{pmatrix} = 0, \quad 1 \leq j \leq mp. \qquad (2)$$

L is defined up to a change of the basis in the subspace. We can normalize to make the rightmost $m \times m$ submatrix of L the unit matrix, so that $L = (K, I)$. Then the determinant in (2) is the Wronskian determinant $W(f_1, \ldots, f_p)$ of the polynomials

$$\begin{array}{rcl} f_1(z) & = & z^{m+p-1} - k_{1,1}z^{m-1} - \ldots - k_{m,1}, \\ f_2(z) & = & z^{m+p-2} - k_{1,2}z^{m-1} - \ldots - k_{m,2}, \\ \ldots & = & \ldots \\ f_p(z) & = & z^m - k_{1,p}z^{m-1} - \ldots - k_{m,p}. \end{array}$$

where $K = (k_{i,j})$.

The Wronski map

The set of p-subspaces in the space of all polynomials of degree $m+p-1$ is parametrized by the Grassmannian $G(p, m+p)$ and the map

$$W : G(p, m+p) \to \mathbf{P}^{mp}$$

is defined by taking the Wronskian determinant of p polynomials.

Change of a basis in the p-space results in multiplying their Wronskian by a constant. The Wronskians, (polynomials of degree mp) considered up to a constant factor form the projective space $\mathbf{P}^{mp}$ which is the target of W.

MTV Theorem can be now restated as follows: *A space of polynomials whose Wronskian has all real roots has a real basis. Preimage* $W^{-1}(H)$ *of the set* $H \subset \mathbf{P}^{mp}$ *of the polynomials whose all roots are real belongs to the real Grassmannian* $G_{\mathbf{R}}(p, m+p)$.

5. It follows from MTV that the image of the *real* Wronski map

$$W_{\mathbf{R}} : G_{\mathbf{R}}(p, m+p) \to \mathbf{RP}^{mp}$$

contains all real polynomials whose all roots are real. Does it contain *all* real polynomials? The answer depends on m and p.

Theorem (EG). *If $m+p$ is odd, then $W_{\mathbf{R}}$ is surjective. Moreover, it has a non-zero topological degree $I(m,p)$, equal to the number of shifted standard Young tableaux (SSYT) with $(m+p-1)/2$ cells in the top row, $(m-p+1)/2$ cells in the bottom row, and of height p, where we assume wlog $m \geq p$.*

1	2	4	5
	3	6	8
		7	9

An explicit expression for $I(m,p)$ (Thrall, 1952):

$$\frac{1!2!\cdots(p-1)!(pm/2)!}{(m-p+2)!(m-p+4)!\cdots(m+p-2)!}$$
$$\times\frac{(m-1)!(m-2)!\cdots(m-p+1)!}{\left(\frac{m-p+1}{2}\right)!\left(\frac{m-p+3}{2}\right)!\cdots\left(\frac{m+p-1}{2}\right)!},$$

This expression gives a lower estimate for the number of real preimages of a real polynomial under the Wronski map.

Theorem (EG). *If both m and p are even, then $W_{\mathbf{R}}$ is not surjective.*

The case when both m and p are odd remains open.

6. To prove the result for odd $m+p$, we consider the special case when the real zeros of the Wronskian satisfy

$$z_1 \gg z_2 \gg \ldots \gg z_{mp} > 0.$$

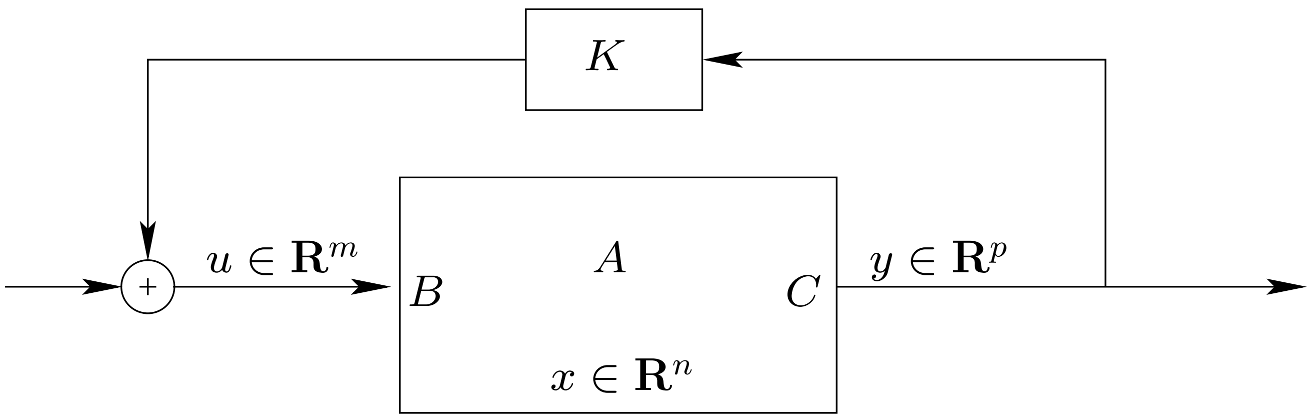
This configuration was used by Sottile in 1999 to prove that many Schubert problems, including the one in the B. & M. Shapiro conjecture *can* have all solutions real.

For such special configurations we compute the topological degree of $W_{\mathbf{R}}$ as the number of SYT with the rectangular shape $m \times p$, counted with the signs depending on the number of inversions. The result is identified with the number of SSYT using a purely combinatorial result of D. White. This gives an explicit formula for the degree.

Both the domain and the target are non-orientable but the degree can be properly defined by passing to orientable coverings.

To prove the non-surjectivity for both m and p even, we use certain very special, degenerate configurations of zeros of the Wronskian, for which we can compute the preimage of the Wronski map explicitly. The zeros of the Wronskian are located on a circle orthogonal to the real line, symmetrically with respect to the real line.

7. Control of a linear system by static output feedback.



$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \\ u &= Ky. \end{aligned}$$

Elimination gives

$$\dot{x} = (A + BKC)x.$$

Pole placement Problem: *given real* A, B, C *and a real polynomial* q *of degree* n*, find real* K*, so that*

$$\det(\lambda I - A - BKC) = q(\lambda).$$

Using:

a) A coprime factorization

$$C(zI - A)^{-1}B = D(z)^{-1}N(z),$$

$$\det D(z) = \det(zI - A),$$

b) The identity

$$\det(I + PQ) = \det(I + QP),$$

we rewrite the pole placement map as

$$\begin{aligned}
&\psi_K(z) = \det(zI - A - BKC) \\
&= \det(zI - A)\det(I - (zI - A)^{-1}BKC) \\
&= \det(zI - A)\det(I - C(zI - A)^{-1}BK) \\
&= \det D(z)\det(I - D(z)^{-1}N(z)K) \\
&= \det(D(z) - N(z)K) \\
&= \begin{vmatrix} D(z) & N(z) \\ K & I \end{vmatrix},
\end{aligned}$$

linear wrt Plücker coordinates. So the pole placement map is a projection of a Grassmann variety.

The Wronski system is defined by $[D(z), N(z)] = E(z)$, the matrix with the rows $F^{(k)}(z)$.

Then $\psi_K(z)$ is the Wronski determinant of

$$\begin{array}{rcl} f_1(z) & = & z^{m+p-1} - k_{1,1}z^{m-1} - \ldots - k_{m,1}, \\ f_2(z) & = & z^{m+p-2} - k_{1,2}z^{m-1} - \ldots - k_{m,2}, \\ \ldots & = & \ldots \\ f_p(z) & = & z^m - k_{1,p}z^{m-1} - \ldots - k_{m,p}. \end{array}$$

where $K = (k_{i,j})$.

Theorem. (EG) *If $m + p$ is odd, and $n = mp$, the real pole placement map is surjective for systems with m inputs, p outputs and state of dimension n in a neighborhood of the Wronski system.*

Theorem. (EG) *If both m and p are even, $q = z(z^2 + 1)^{mp/2-1}$ is not covered by the real pole placement map for an open set of systems with m inputs, p outputs and state of dimension mp.*

8. Instead of subspaces osculating the rational normal curve F one can consider subspaces *spanned* by the points of F as the Schubert data. This leads to the following

Secant Conjecture. *Consider sets*

$$A_j = \{z_{j,1}, \ldots, z_{j,p}\}, \quad 1 \le j \le mp,$$

on the real projective line $\mathbf{RP}^1$*, and suppose that these sets are separated, that is belong to some disjoint intervals* $I_j \subset \mathbf{RP}^1$*. Let* X_j *be the linear* p*-subspace in* $\mathbf{R}^{m+p}$ *spanned by* $F(z_{j,k})$, $1 \le k \le p$*. Then each* m*-space intersecting all* p*-spaces* X_j *is real.*

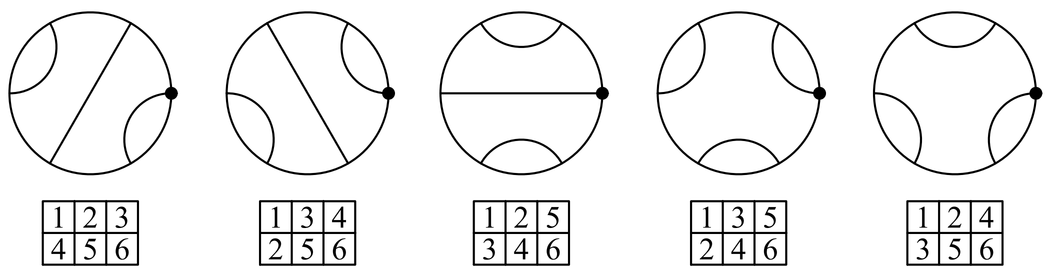
This is known for $p = 2$ (EGSV: Eremenko, Gabrielov, M. Shapiro, Vainshtein).

Mukhin, Tarasov and Varchenko have a partial result for every m and p, with strong additional assumptions on the points $z_{j,k}$.

9. The method in EGSV is based on the earlier approach of Eremenko and Gabrielov to the original Shapiro conjecture.

Consider the Wronski map for $p = 2$, $(f_1, f_2) \mapsto W(f_1, f_2)$. Zeros of the Wronskian are critical points of the rational function $f = f_1/f_2$. Thus finding a 2-space of polynomials with the given Wronskian is equivalent to finding a rational function with prescribed critical points.

To each real rational function f whose all critical points are real we assign a topological object which we call a *net*. The net of f is essentially $f^{-1}(\mathbf{RP}^1)$, with one distinguished critical point, modulo orientation-preserving homeomorphisms of $\mathbf{CP}^1$ preserving $\mathbf{RP}^1$ and leaving the distinguished point fixed. The number of different nets equals the complex degree of the Wronski map for $p = 2$.



All nets for $d = 4$

When the zeros of the Wronskian satisfy

$$z_1 \gg z_2 \gg \ldots \gg z_{2m} > 0,$$

the full preimage of the Wronski map can be studied by asymptotic analysis, and all possible nets occur in this preimage. As the (distinct) critical points move continuously from this position, the preimages of the Wronski map cannot collide since they all have different nets! So they must remain real.

10. To prove the Secant conjecture for $p = 2$ we restate it as a problem about rational functions, using the same Wronski map. The equivalent statement is:

EGSV Theorem. *Consider pairs*

$$A_j = \{z_{j,1}, z_{j,2}\}, \quad 1 \le j \le 2d - 2$$

on the real projective line $\mathbf{RP}^1$, *and suppose that these pairs are separated. Then any rational function of degree* d *which satisfies*

$$f(z_{j,1}) = f(z_{j,2}) \quad j = 1, \ldots, 2d - 2$$

is equivalent to a real rational function by post-composition with a fractional-linear transformation.

This is a special case of the theorem in EGSV, which allows $|A_j| = a_j + 1, 1 \le j \le k$ where $1 \le a_j \le d - 1$ and

$$\sum_k a_j = 2d - 2.$$

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To prove EGSV Theorem, we choose disjoint closed intervals $I_j \supset \{z_{j,1}, z_{j,2}\}$ place a point t_j on each interval I_j and consider the real rational function f degree d with critical points t_j. If we leave all critical points except one, t_k fixed, and let t_k run over I_k, we easily see that there is a position of t_k which gives $f(z_{k,1}) = f(z_{k,2})$. This is because at the extreme positions of t_k, $f|_{I_k}$ is monotone in the opposite directions. Then the proof is concluded with a well-known topological lemma.

11. If the groups of points in the Secant Conjecture are not separated, some solution subspaces can be non-real, as computations of Sottile's team show. Nevertheless, it was discovered in these computations that for certain topological restrictions on the groups of points, there are *lower bounds* on the numbers of real solutions.

In some special cases for $p = 2$ existence of such lower bounds was proved by Azar and Gabrielov. The nets of rational functions are used in the proof.

Suppose that $2d-3$ real critical points t_j of a rational function f of degree d are prescribed, and in addition it is required that $f(z_1)=f(z_2)$ at some real points z_1, z_2. Let M be the number of points t_k on an arc between z_1 and z_2. Lower bounds for the number of real solutions in terms of d and M are obtained.

To do this, a family of real rational functions with $2d-3$ prescribed real critical points and one movable real critical point is considered. As the movable critical point rotates around $\mathbf{RP}^1$ several times, until the function returns to the original one, there are several positions of this movable critical point where the equality $f(z_1)=f(z_2)$ must be attained, due to the inequalities on the increment of the argument of f on the arc between (z_1 and z_2).