MATH 511, First exam, solutions. Spring 2002

1. Consider the matrix equation $PA = LU$, where

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) What is the rank of A? Ans: 2.

b) Find a basis in the null space of A. Ans: one basis is

$$(0, 2, 0, -2, 1)^T,\ (0, -2, 1, 0, 0)^T,\ (1, 0, 0, 0, 0)^T.$$

c) Find a basis in the column space of A. Ans: for example,

$$(1, 0, 1)^T,\ (3, 1, 4)^T.$$

2. Tell whether this matrices are invertible, and if yes, find the inverse.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Ans: Yes; everyone found the inverse correctly.

3. Let R_1 and R_2 be two reflections of the plane, R_1 with respect to the $x-$axis, and R_2 with respect to the line $y = x \tan 20°$. Write the matrix of the transformation $R = R_1 R_2$ and find its geometric meaning: is it a rotation, a reflection, a projection, or none of the above.

Ans: It is the rotation by 40° clockwise.

In general, if you perform several rotations and several reflections, in any order, the result is a rotation if the number of reflections was even, and a reflection if it was odd. Prove this!

4. Find a basis in the space of all vectors in $\mathbf{R^6}$ that satisfy $x_1 + x_2 = x_3 + x_4 = x_5 + x_6$.

Ans: For example,

$$(1,0,1,0,0,1)^T, (1,0,1,0,1,0)^T, (0,0,-1,1,0,0)^T, (-1,1,0,0,0,0)^T.$$

You just had to write the system of two or three equations, and apply the standard row operation process of finding a basis in a nullspace of a matrix.

5. How many permutation matrices 5×5 are there? Do they span the space of all 5×5 matrices? Are they linearly independent?

Ans: a) 120, b) No, c) No.

The answer to c) is no for an evident reason: the dimension of the space of all 5×5 matrices is only 25, so it cannot contain 120 linearly independent vectors.

The answer to b) is a bit harder to justify. One has to notice that all permutation matrices have the following Property: each column has the same sum of elements (because this sum is 1 for each column). Now this Property defines a linear subspace in the space of all matrices: every linear combination of matrices with this Property also has this Property. Evidently there are 5×5 matrices, not having this Property. So permutation matrices do not span the space of all matrices.

Remark. Several students misunderstood the definition of a permutation matrix. It is an ARBITRARY permutation of rows of the unit matrix, NOT JUST an exchange of two rows. A product of two row exchanges is not in general a row exchange. But a product of two permutation matrices is always a permutation matrix.

I realize that the book is extremally fuzzy on this, (as well as in many other places). But I explained this point in class. Still, after some hesitation, I gave a small partial credit to those who said that there are 11 permutation matrices (they counted only row exchanges), and THEN MADE A RIGHT CONCLUSION FROM THIS.

6. Let V be the vector space consisting of all linear transformations from $\mathbf{R}^n$ to $\mathbf{R}^n$ where addition of linear transformations is defined by the rule $(L_1+L_2)(x) = L_1(x)+L_2(x)$ and multiplication by numbers by the rule $(cL)(x) = cL(x)$. Find the dimension of V.

Ans: n^2 because such linear transformations are represented by matrices, the correspondence between linear transformations and matrices is one-to-one (once a basis is fixed), and sum of linear transformations is defined in such a way that it corresponds to the sum of matrices, and similarly with product with numbers. So the space of linear transformations “looks the same way” as the space of matrices. Mathematicians say that such spaces are ISOMORPHIC. Evidently, isomorphic spaces have the same dimension, because a basis in one of them corresponds to a basis in another one.

7. True or false (1 point for each correct answer, no explanation necessary):

a) If the number of unknowns in the system $Ax = b$ is greater than the number of equations, then it has a solution. False. For example $0x_1 + 0x_2 = 1$. One equation, two unknowns, no solutions.

b) If 10 vectors span a vector space then this space has dimension 10. False.

c) If a square matrix has linearly independent columns then so does A^2. True. Such matrix is non-singular (=invertible). Then A^2 is also non-singular.

d) If a space is spanned by 11 vectors then every set of 12 vectors in this space is linearly dependent. True.

e) The row space of a matrix has unique basis which can be obtained by reducing this matrix to the row echelon form. False as stated. The crucial word which makes it false is ”unique”. Every non-trivial vector space (over the real numbers) has infinitely many different bases.

f) Every subspace of $\mathbf{R}^n$ is the row space of some matrix. True.

g) Suppose that B is a basis of a vector space V. Then for every subspace V', one can discard some vectors of B to obtain a basis of V'. False. For example, e_1 and e_2 form a basis in R^2, but none of these vectors belongs to the subspace spanned by $e_1 + e_2$ (also called the “line $x = y$”). So you canot make a basis for this line of vectors e_1 and e_2.

h) There is a matrix whose rowspace contains the vector $(1, 1, 1, 1)$ and nullspace contains the vector $(1, 2, 3, 4)^T$. False.

$$1 \times 1 + 1 \times 2 + 1 \times 3 + 1 \times 4 \neq 0.$$

i) The nullspace of every invertible matrix is trivial. True.

j) Every square matrix with linearly dependent columns is singular. True.