Landau's constant L is defined in the following way. Let f be a function analytic in the unit disc, $f(0) = 1$. Let $L(f)$ be the radius of the largest disc in the image of f. Then $L = \inf_f L(f)$, where the inf is taken over all such functions.

Problem. *Following the line of Ahlfors's proof of the Bloch theorem, prove that* $L \geq 1/2$.

Hint: use the ultrahyperbolic metric $\lambda(z)|dz|$, which is the pullback of $\rho(w)|dw|$ with

$$\rho = \frac{c}{R|\log R|},$$

where R is the distance from z to the boundary of the image of the unit disc under f, and c is an appropriately chosen constant.

Remarks. The upper estimate and the conjectured exact value of the Landau constant is

$$\Gamma(1/3)\Gamma(5/6)/\Gamma(1/6) = 0.5432589\ldots.$$

The conjectured extremal f is the universal cover of the plane minus a hexagonal lattice.

The current world record in the lower estimate is $L > 1/2 + 10^{-335}$ which is due to Yanagihara (1995).