24. Let $\rho(z)|dz|$ be the hyperbolic metric on some hyperbolic Riemann surface. Prove that the density ρ satisfies the differential equation

$$\Delta \log \rho = 4\rho^2.$$

Do it in two steps:

a) Verify this by explicit calculation for the case of the unit disc.

b) Show that the differential equation is invariant under any conformal mapping, that is if $w(z)$ is a conformal map and

$$\rho_1(w)|dw| = \rho_1(w(z))|w'(z)||dz| = \rho(z)|dz|,$$

then

$$\frac{\Delta \log \rho_1}{\rho_1^2} = \frac{\Delta \log \rho}{\rho^2}.$$

So this ratio is a well defined function on the Riemann surface (does not depend on the local coordinate z). The negative of this ratio is called the *curvature* of the metric. Thus the hyperbolic metric has curvature -4.

In many books they define the hyperbolic metric in the disc by the length element $2|dz|/(1-|z|^2)$ so that it has curvature -1.

If you have difficulty with this exercise, consult Ahlfors, Conformal Invariants (on reserve in the Math library for this course).