Lecture 11.1

Reading: Sections 1 and 3 of Ch. 12

Practice: 12.3, 12.20, 12.23, 12.26

Homework: ~~12.~~ 12-1, 12-4, 12.18

~~De~~ Tensor products of vector spaces

Def Let $V_1, \dots, V_k, W$ be vector spaces. A map $F: V_1 \times V_2 \times \dots \times V_k \to W$ is <u>k-multilinear</u> if it is linear in each factor.

<u>Examples</u> Dot product, crossproduct on $\mathbb{R}^3$, determinant, Lie bracket.

Def Set of all k-multilinear maps $V_1 \times \dots \times V_k \to W$ denoted $L(V_1, \dots, V_k ; W)$.

Have a map

$$L(V_1, \dots, V_k ; \mathbb{R}) \times L(W_1, \dots, W_l ; \mathbb{R}) \to L(V_1, \dots, V_k, W_1, \dots, W_l ; \mathbb{R})$$

$$(F, G) \mapsto F \otimes G$$

where

$$F \otimes G(v_1, \dots, v_k, w_1, \dots, w_l) = F(v_1, \dots, v_k) \, G(w_1, \dots, w_l)$$

multiplication in $\mathbb{R}$.

Note that $(F \otimes G) \otimes H = F \otimes (G \otimes H)$ for $\mathbb{R}$-valued multilinear maps F, G, H.

Prop 12.4 Let $V_1, \dots, V_k$ have dimensions $n_1, \dots, n_k$, and consider bases $(E_1^{(j)}, \dots, E_{n_j}^{(j)})$ for V_j, $j = 1, \dots, k$. If $(\varepsilon^1_{(j)}, \dots, \varepsilon^{n_k}_{(j)})$ denote the dual bases, then

$$\{ \varepsilon^{i_1}_{(1)} \otimes \cdots \otimes \varepsilon^{i_k}_{(k)} \mid 1 \leq i_1 \leq n_1, \dots, 1 \leq i_k \leq n_k \}$$

is a basis for $L(V_1, \dots, V_k; \mathbb{R})$.

In particular,

$$\dim L(V_1, \dots, V_k; \mathbb{R}) = \prod_{i=1}^{k} \dim V_i .$$

Now we generalize/abstract.

Def If S is any set, let $\mathcal{F}(S)$ be the (real) vector space freely generated by S.

E.g. $\mathcal{F}(\emptyset) \cong \{0\}$, $\mathcal{F}(\{x\}) \cong \mathbb{R}$

Def Let $V_1, \dots, V_k$ be real vector spaces. Define $\mathcal{R} \leq \mathcal{F}(V_1 \times \cdots \times V_k)$ by taking the span of all elements of form

$(v_1, \dots, a v_i, \dots, v_k) - a(v_1, \dots, v_i, \dots, v_k)$

and $(v_1, \dots, v_i + v_i', \dots, v_k) - (v_1, \dots, v_i, \dots, v_k) - (v_1, \dots, v_i', \dots v_k)$.

The <u>tensor product of $V_1, \dots, V_k$</u> is the vector space

$$V_1 \otimes \cdots \otimes V_k := \mathcal{F}(V_1 \times \cdots \times V_k)/\mathcal{R},$$

together with the <u>natural projection map</u>

$$\pi : V_1 \times \cdots \times V_k \to V_1 \otimes \cdots \otimes V_k,$$

(which is k-multilinear). Given vectors $v_1 \in V_1, \dots, v_k \in V_k$, denote $\pi(v_1, \dots, v_k)$ by $v_1 \otimes \cdots \otimes v_k$.

Why do this?! The tensor product vector space is the "universal" vector space that allows us to convert questions of multilinearity to questions of "vanilla" linearity.

<u>Prop 12.7</u> For any multilinear map

$$A : V_1 \times \cdots \times V_k \to W,$$

there exists a <u>unique</u> <u>linear</u> map

$$\tilde{A} : V_1 \otimes \cdots \otimes V_k \to W,$$

such that get commuting diagram

$$\begin{tikzcd} V_1 \times \cdots \times V_k \arrow[r, "A"] \arrow[d, "\pi"] & W \\ V_1 \otimes \cdots \otimes V_k \arrow[ur, "\tilde{A}"'] & \end{tikzcd}$$

<u>Prop 11.8</u> $\dim V_1 \otimes \cdots \otimes V_k = \prod_{i=1}^{k} \dim V_i$

Prop 12.9 There are *natural* isomorphisms

$$V_1 \otimes (V_2 \otimes V_3) \cong (V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes V_2 \otimes V_3$$

$$v_1 \otimes (v_2 \otimes v_3) \mapsto (v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes v_2 \otimes v_3$$

Prop 12.10 There is a *natural* isomorphism

$$V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \ldots, V_k; \mathbb{R}).$$

"natural" means we can treat as "=", not just "≅".

Co- and contra-variant tensors

Def A *covariant k-tensor* on V is an element of

$$T^k(V^*) := \underbrace{V^* \otimes \cdots \otimes V^*}_{k} \cong L(\underbrace{V, \ldots, V}_{k}; \mathbb{R})$$

Ex $T^0(V^*) = \mathbb{R}$ $\qquad T^2(V^*) = L(V, V; \mathbb{R})$

"bilinear forms on V"

$T^1(V^*) = V^*$

Def A *contravariant l-tensor* on V is an element of

$$T^k(V) = V \otimes \cdots \otimes V \cong L(V^*, \ldots, V^*; \mathbb{R})$$

Can also define "mixed" tensors.

Tensor fields

Def For a smooth manifold M, have several types of "tensor bundles":

$$T^k T^* M := \bigsqcup_{p \in M} T^k(T_p^* M) \quad \text{(covariant } k\text{-tensors on } M)$$

$$T^k T M := \bigsqcup_{p \in M} T^k(T_p M) \quad \text{(contravariant } k\text{-tensors on } M)$$

+ mixed versions.

A section of one of these bundles is generally called a <u>tensor field on M</u>.

Suppose, e.g., $A \in \Gamma(T^k T^* M)$, and pick coordinates (x^i) on M. Then there exist smooth $\mathbb{R}$-valued functions $A_{i_1, \dots, i_k}$ such that

$$A = \underbrace{A_{i_1, \dots, i_k}}_{\text{"component functions" of the covariant vector field } A.} dx^{i_1} \otimes \cdots \otimes dx^{i_k}.$$

There are several smoothness criteria you should read about.

Pullbacks

Def Let $F: M \to N$ be smooth and $A \in \Gamma(T^k T^* N)$ a covariant k-tensor field on N. The pullback F^*A is the tensor field on M defined by

$$(F^*A)_p := dF_p^*(A_{F(p)})$$

where $dF_p^*(A_{F(p)})(v_1,\dots,v_m) = A_{F(p)}(dF_p v_1,\dots,dF_p v_m)$

Prop 12.25 F^*A smooth. Also:

(a) $F^*(fA) = (f\circ F)\, F^*A$

(b) $F^*(A\otimes B) = F^*(A)\otimes F^*(B)$

(c) $F^*(A+B) = F^*A + F^*B$

(d) $F^*G^*A = (G\circ F)^*A$

(e) $Id^*A = A$.

Corollary In coordinates,

$$F^*(B_{i_1,\dots,i_k}\, dy^{i_1}\otimes\cdots\otimes dy^{i_k})$$

$$= B_{i_1,\dots,i_k}\circ F\, d(y^{i_1}\circ F)\otimes\cdots\otimes d(y^{i_k}\circ F).$$

Comment at end of class: most tensors not too interesting. Next time: alternating tensors.

Symmetric tensors

Def A covariant k-tensor α on V is <u>symmetric</u> if

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = \alpha(v_1, \dots, v_j, \dots, v_i, \dots v_k)$$

for all $v_1, \dots, v_k \in V$ and $1 \le i \le j \le k$.

Def Let S_k be symmetric group on $\{1, \dots, k\}$. For any covariant k-tensor α and $\sigma \in S_k$, get another covariant k-tensor ${}^{\sigma}\alpha$ defined as

$${}^{\sigma}\alpha(v_1, \dots, v_k) := \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Yields a representation

$$S_k \longrightarrow GL(T^k V^*),$$

i.e. ${}^{\sigma}({}^{\tau}\alpha) = {}^{(\sigma\tau)}\alpha$, and ${}^{1}\alpha = \alpha$ for $1 \in S_k$ identity element.

<u>Note</u>: α is symmetric iff ${}^{\sigma}\alpha = \alpha$ $\forall \sigma \in S_k$ (Why?)

Def $\Sigma^k(V^*) \le T^k(V^*)$ is subspace of symmetric tensors. (Why a subspace?)

Have a projection operator

$$\mathrm{Sym}: T^k(V^*) \to \Sigma^k(V^*) \subseteq T^k(V^*)$$

defined by

$$\mathrm{Sym}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \alpha.$$

<u>Prop 12.14</u> $\mathrm{Sym}(\alpha)$ is symmetric and $\mathrm{Sym}(\alpha) = \alpha$ iff α symmetric already.

<u>Def</u> a <u>symmetric (covariant) tensor field</u> on M is a section α of $T^k T^*M$ such $\alpha_p \in T^k T^*_p M$ is symmetric for all $p \in M$.

<u>Alternating Tensors</u>

Recall $\mathrm{sgn}: S_k \to \{\pm 1\}$

$$\sigma \mapsto \begin{cases} +1 & \text{if } \sigma \text{ a product of even \# tran} \\ -1 & \text{else.} \end{cases}$$

<u>Def</u> a tensor $\alpha \in T^k(V^*)$ is <u>alternating</u> (or anti-symmetric) if