Lecture 11.3

Reading: First section of Ch. 14

Practice: 14.12, 14.14

Homework: 14-1

I'm approaching this material differently from Lee, to highlight formal properties rather than calculations. (Even though both important!)

Exterior Algebras of Vector Spaces

Recall def'n of alternating covariant k-tensors.
Aka: exterior forms, multi-covectors, or k-covectors.

Lemma 14.1 Let $\alpha \in T^k V^*$. TFAE:

(a) α alternating, i.e. $\alpha \in \Lambda^k V^*$

(b) $\alpha(v_1, \dots, v_k) = 0$ for linearly dependent k-tuples $v_1, \dots, v_k \in V$

(c) $\alpha(\dots, w, \dots, w, \dots) = 0$, i.e. α vanishes whenever two arguments are equal.

Proof (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c) easy. Let's do (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b). $\square$

Def Let $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$. The wedge product (or exterior product) of ω and η is

$$\omega \wedge \eta := \frac{(k+l)!}{k!\, l!} \operatorname{Alt}(\omega \otimes \eta) \in \Lambda^{k+l}(V^*).$$

Prop 14.11 Wedge product is bilinear, associative, and graded commutative (or super commutative).
This latter means

$$\eta \wedge \omega = (-1)^{kl}\, \omega \wedge \eta.$$ In particular, $\omega \wedge \omega = 0$.

Proof Bilinearity immediate, b/c $\otimes$ bilinear and Alt linear. Associativity is a bit of a slog, but let's do it... assume $l=1$. Then use induction $\square$

Def The exterior algebra of V w/ $\dim V = n$ is the vector space

$$\Lambda(V^*) := \bigoplus_{k=0}^{n} \Lambda^k(V^*)$$

equipped with the wedge product. (aka "Grassmann algebra")

Note Since $\dim V = n$, $\Lambda^k(V^*) = 0$ for $k > n$. Why? (Prop 14.11(b)).

We'll show momentarily that

$$\dim \Lambda^k(V^*) = \binom{n}{k}$$

hence,

$$\dim \Lambda(V^*) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

First, I want to introduce another type of operation of $\Lambda(V^*)$.

Def Let $v \in V$. The interior product (or

Contraction along v) is the linear map

$$i_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$$

with

$$(i_v \omega)(w_1, \dots, w_{k-1}) := \omega(v, w_1, \dots, w_{k-1}).$$

Also written $v \lrcorner \omega$. "v into ω"

Lemma 14.13 (a) $i_v \circ i_v = 0$

(b) i_v satisfies a graded product rule (or "super Liebniz rule"):

"super derivation"

$$i_v(\omega \wedge \eta) = (i_v \omega) \wedge \eta + (-1)^k \omega \wedge (i_v \eta)$$

Proof (a) Follows from Prop 14.11(b).

(b) We'll prove momentarily that $\Lambda^k(V^*)$ is spanned by decomposable k-covectors of the form

$$\omega^1 \wedge \dots \wedge \omega^k \quad \text{for} \quad \omega^i \in \Lambda^1(V^*) = V^*.$$

Thus, it suffices to prove the identity for decomposable ω and η. We're going to use cofactor expansion. (!)

We can use induction on k. The base case $k=1$ is most important.

$$\cancel{i_v(\omega_1 \wedge \dots \wedge \omega_k \wedge \eta_1 \wedge \dots \wedge \eta_l)(v_1, \dots, v_{k+l-1})}$$
$$= (\omega_1 \wedge \dots \wedge \omega_k \wedge \eta_1 \wedge \dots \wedge \eta_l)(v, v_1, \dots, v_{k+l-1})$$

$$\boxed{\begin{array}{l}\text{Claim (to be proved momentarily)} \\ (\omega^1 \wedge \dots \wedge \omega^k)(v_1, \dots, v_k) = \det \omega^j(v_i)\end{array}}$$

We'll write $V_1 = v$ and assume $w^1 = w$ and $w^2 \wedge \cdots \wedge w^{\ell+1} = \eta$.

Then $i_v(w \wedge \eta) = i_v(w^1 \wedge w^2 \wedge \cdots \wedge w^{\ell+1})$

$$i_v(w \wedge \eta)(v_2, \ldots, v_{\ell+1}) = w^1 \wedge \cdots \wedge w^{\ell+1}(v_1, v_2, \ldots, v_{\ell+1})$$

$$= \det w^j(v_i)$$

$$= \sum_{i=1}^{\ell+1} (-1)^{i-1} w^i(v_1) \det V_1^i$$

where $\det V_1^i$ is $(w^j(v_i))_{i,j}$ with 1^{st} column and i^{th} row removed. Thus $\det V_1^1 = (w_2 \wedge \cdots \wedge w_{\ell+1})(v_2, \ldots, v_{\ell+1})$ and we have

$$i_v(w \wedge \eta)(v_2, \ldots, v_{\ell+1}) = w^1(v_1)(w_2 \wedge \cdots \wedge w_{\ell+1})(v_2, \ldots, v_{\ell+1}) + \sum_{i=2}^{\ell+1} (-1)^{i-1} w^i(v_1) \det V_1^i$$

$$= (i_v w) \wedge \eta(v_2, \ldots, v_{\ell+1}) + (-1)^1 w \wedge (i_v \eta)(v_2, \ldots, v_{\ell+1}).$$

need to stare hard!

$\square$