Lecture 12.1

Reading: Ch. 14

Practice: 14.4, 14.6, 14.14, 14.17, 14.22, 14.25

Homework: 14-9

Elementary k-covectors

To prove some identities from last time, need good "coordinates" on $\Lambda(V^*)$.

Warm-up If $\dim V = n$, then $\dim \Lambda^n(V^*) = 1$, spanned by "det" with respect to any basis of V.

Proof Let $\alpha \in \Lambda^n(V^*)$. Because it is alternating, it is entirely determined by $\alpha(E_1, \dots, E_n) \in \mathbb{R}$, where $E_1, \dots, E_n$ a basis of V. ~~If $\varepsilon^1, \dots, \varepsilon^n$ dual basis,~~ then Let

$$\det_{E_1, \dots, E_n} \in \Lambda^n(V^*)$$

be defined by $\det_{E_1, \dots, E_n}(v_1, \dots, v_n) = \det(\varepsilon^j(v_i))$, where $\varepsilon^1, \dots, \varepsilon^n$ the dual basis. Then $\alpha = \alpha(E_1, \dots, E_n) \det_{E_1, \dots, E_n}$. $\square$

Def Let $I = (i_1, \dots, i_k)$ be a multi-index of length k such that $1 \le i_j \le n$, and fix a basis $\varepsilon^1, \dots, \varepsilon^n$ of V^*. The elementary alternating tensor ε^I def'd as

$$\varepsilon^I(v_1, \dots, v_k) := \det \begin{pmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & & \vdots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{pmatrix}$$

For convenience, define

$$\delta^I_J = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k} \\ \vdots & & \vdots \\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_k} \end{pmatrix}$$

$$= \begin{cases} \operatorname{sgn} \sigma & \text{if } I, J \text{ have no repeated index and } I = \sigma J. \\ 0 & \text{else} \end{cases}$$

__Lemma 14.7__

(a) If I has a repeated index, $\varepsilon^I = 0$

(b) If $J = {}^\sigma I$, then $\varepsilon^I = (\operatorname{sgn} \sigma)\, \varepsilon^J$

(c) $\varepsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta^I_J$.

A multi-index $I = (i_1, \dots, i_k)$ is ~~increasing~~ if

$i_1 < i_2 < \dots < i_k$.

__Prop 14.8__ Let

$\mathcal{E} := \{ \varepsilon^I \mid I$ increasing of length $k \}$.
$\quad$ w/ $1 \le i_j \le n$

Then $\mathcal{E}$ is a basis for $\Lambda^k(V^*)$. In particular

$$\dim \Lambda^k(V^*) = \binom{n}{k}.$$

Where does "determinant of Jacobian" come from when integrating?

Prop 14.9 If $\dim V = n$ and $T: V \to V$ linear, then for all $\omega \in \Lambda^n(V^*)$,

$$\omega(Tv_1, \dots, Tv_n) = (\det T)\,\omega(v_1, \dots, v_n).$$

Lemma 14.10

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}$$

where IJ is concatenation of I and J.

Prop 14.11

(d) $\varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_k} = \varepsilon^I$

(e) $\omega^1 \wedge \dots \wedge \omega^k (v_1, \dots, v_k) = \det(\omega^j(v_i))$.

Differential forms

$\Lambda^k T^*M$ is the sub-bundle of $T^k T^*M$ consisting of alternating tensors.

Ex 14.14 $\Lambda^k T^*M$ is smooth subbundle of rank $\binom{n}{k}$. [Hint: show $\mathrm{Alt}: T^k T^*M \to T^k T^*M$ is well-def'd, smooth, and has constant rank. Then $\Lambda^k T^*M = \mathrm{Im}\,\mathrm{Alt}$.]

Def A section of $\Lambda^k T^*M$ is called a <u>differential form</u> of degree k.

$$\Omega^k(M) := \Gamma(\Lambda^k T^*M).$$ If ω a k-form, η an l-form,

Define $\omega \wedge \eta$ pointwise by

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p \in \Omega^{k+l}(M).$$

If $F \in \Lambda^0 T^*M := C^\infty(M)$, we define

$$F \wedge \omega = F\omega.$$

Wedge product makes

$$\Omega^*(M) := \bigoplus_{k=0}^{n} \Omega^k(M)$$

into an associative, supercommutative graded <u>algebra</u> module over $C^\infty(M)$.

In coordinates, each $\omega \in \Omega^k(M)$ looks like

$$\omega = \omega_I \, dx^I$$

where $\omega_I \in C^\infty(M)$. determined by

$$\omega_I(p) = \omega\left(\frac{\partial}{\partial x^{i_1}}\Big|_p, \dots, \frac{\partial}{\partial x^{i_k}}\Big|_p\right).$$

Can pullback forms just like any covariant vector fields. Easy to check a pullback of something alternating is again alternating.

$$(F^*\omega)_p(v_1,\dots,v_k) = \omega_{F(p)}(dF_p v_1,\dots, dF_p v_k).$$

Lemma 14.16 $F: M \to N$ C^∞

(a) $F^*: \Omega^k(N) \to \Omega^k(M)$ linear

(b) $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$

(c) $F^*(\omega_I \, dy^I) = \omega_I \circ F \, d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F)$.

Ex 14.17 Prove this. (Hint: (a) immediate by above comment. (c) follows from (b) and what we already know for 1-forms. For (b), show that for any $\alpha \in \Gamma(T^k T^*M)$, and any $\sigma \in S_k$, $F^*({}^\sigma\alpha) = {}^\sigma F^*(\alpha)$.)

Prop 14.20 Let $F: M \to N$ be smooth, $\dim M = \dim N = n$. Let (x^i), (y^i) be coordinates on $U \in M$, $V \in N$, resp. Then

$$F^*(dy^1 \wedge \dots \wedge dy^n) = \det DF \, dx^1 \wedge \dots \wedge dx^n$$

where DF is Jacobian matrix in coordinates.

Proof

$$F^*(dy^1\wedge\cdots\wedge dy^n) = dF^1\wedge\cdots\wedge dF^n.$$

$$(dF^1\wedge\cdots\wedge dF^n)\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right) = \det\left(dF^j\left(\frac{\partial}{\partial x^i}\right)\right) = \det\left(\frac{\partial F^j}{\partial x^i}\right)$$

$$= \det DF\left[(dx^1\wedge\cdots\wedge dx^n)\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right)\right]$$

$$\Rightarrow F^*(dy^1\wedge\cdots\wedge dy^n) = \det DF\,(dx^1\wedge\cdots\wedge dx^n). \quad \square$$

Def If $X\in\mathfrak{X}^*(M)$ and $\omega\in\Omega^k(M)$, define $X\lrcorner\,\omega = i_X\omega$ pointwise by

$$(X\lrcorner\,\omega)_p = X_p\lrcorner\,\omega_p.$$

Ex 14-22 $X\lrcorner\,\omega$ is smooth.

$i_X:\Omega^k(M)\to\Omega^{k-1}(M)$ linear over $C^\infty(M)$, hence, corresponds to a bundle map $i_X:\Lambda^k T^*M\to\Lambda^{k-1}T^*M$.