Lecture 12.2

Reading: Section 3 of Ch 14; sections 1+2 of Ch. 15

Practice: 14.25

Homework: 14-9

Exterior Derivative

Def Let $I = (i_1, \dots, i_k)$ and $w_I \in C^\infty(\mathbb{R}^n)$.
Define the exterior derivative of the k-form $w_I \, dx^I$ (no Einstein summation yet) to be the $(k+1)$-form

$$d(w_I \, dx^I) := dw_I \wedge dx^I$$

where dw_I is the differential of w_I (a 1-form).

Prop 14.23

(a) For each k, d extends to a well-defined linear map

$$d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n).$$

note: not $C^\infty(\mathbb{R}^n)$ linear (b/c (b))

(b) If $w \in \Omega^k(\mathbb{R}^n)$ and $\eta \in \Omega^l(\mathbb{R}^n)$, then

$$d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta.$$

(c) $d \circ d = 0$.

(d) d commutes with pullbacks: if $U \subseteq \mathbb{R}^n$, V open and $F: U \to V$ smooth, then $d(F^*\omega) = F^*(d\omega)$ for every $\omega \in \Omega^*(\mathbb{R}^n)$.

Proof

(a) Main (small) issue is that we have defined d on a spanning set of $\Omega^*(\mathbb{R}^n)$ that is not linearly independent. We can easily see that

$$d(r\omega_I\, dx^I) = d(r\omega_I) \wedge dx^I$$
$$= r\, d\omega_I \wedge dx^I = r\, d(\omega_I\, dx^I),$$

however the dx^I are only linearly independent if we consider increasing multiindices I.

However, there is no issue: if $J = \sigma I$,

then $d(\omega_J\, dx^J) = d\omega_J \wedge dx^J$

$$= \text{sgn}\,\sigma\; d\omega_J \wedge dx^I$$

$$= \text{sgn}\,\sigma\; d(\omega_J\, dx^I).$$

(b) Suffices to assume, by (a), that

$\omega = u\,dx^I, \quad \eta = v\,dx^J$. Then

$\wedge$ is $C^\infty(\mathbb{R}^n)$ linear!

$$d(\omega\wedge\eta) = d\left[(u\,dx^I)\wedge(v\,dx^J)\right] = d(uv\,dx^I\wedge dx^J)$$

$$= (u\,dv + v\,du)\wedge dx^I\wedge dx^J$$

$$= u\,dv\wedge dx^{IJ} + v\,du\wedge dx^{IJ}$$

Recall: $dx^I\wedge dx^J = dx^{IJ}$

$$d\omega\wedge\eta + (-1)^k\omega\wedge d\eta$$

$$= (du\wedge dx^I)\wedge(v\,dx^J) + (-1)^k(u\,dx^I)\wedge(dv\wedge dx^J)$$

$$= (-1)^k\,u\,dx^I\wedge dv\wedge dx^J + v\,du\wedge dx^{IJ}$$

$$= u\,dv\wedge dx^{IJ} + v\,du\wedge dx^{IJ}.$$

(c) We'll show $(d\circ d)\omega = 0$ for $\omega\in\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ first, then apply part (b).

$$d(d\omega) = d\left(\frac{\partial\omega}{\partial x^j}\,dx^j\right)$$

$$= \frac{\partial^2\omega}{\partial x^i\partial x^j}\,dx^i\wedge dx^j$$

$$= 0 \quad \text{b/c} \quad \frac{\partial^2\omega}{\partial x^i\partial x^j} = \frac{\partial^2\omega}{\partial x^j\partial x^i}$$

and $dx^i\wedge dx^i = 0$.

Now for $w \in \Omega^k(\mathbb{R}^n)$, ~~[illegible]~~ might as well assume $w = w_I \, dx^I$. Then

$$d^2(w) = d(dw_I \wedge dx^I)$$

$$= (d^2 w_I) \wedge dx^I = 0$$

multiple ways to look at this:

- it follows from def'n of d
- $dx^I = 1\,dx^I$ and $d1 = 0$
- $d(dx^I)$

$$= d(dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k})$$

$$\text{(b)} = \sum_{j=1}^{k} (-1)^{i+1} (dx^{i_1} \wedge \cdots \wedge \underbrace{d(dx^{i_j})}_{0!} \wedge \cdots dx^{i_k})$$

(d) again assume $w = ~~w_I\,dx^I~~$ $u\,dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, for $u \in C^\infty(M)$. Then

$$F^*(d(u\,dx^{i_1} \wedge \cdots \wedge dx^{i_k}))$$

$$= F^*(du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k})$$

$$= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)$$

$$\overset{(c)}{=} d[(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)]$$

$$= d(F^*(u\,dx^{i_1} \wedge \cdots \wedge dx^{i_k}))$$

$\square$

Part (d) is key to being able to define exterior derivative on all smooth manifolds

<u>Theorem 14.24</u> Let M be a smooth manifold. ~~For each~~ There exist unique operators

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

for all k — called the exterior differentiator — satisfying following properties:

(i) d linear over $\mathbb{R}$

(ii) If $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

(iii) $d \circ d = 0$

(iv) $d : \Omega^0(M) = C^\infty(M) \to \Omega^1(M)$ is usual differential defined by $(dF)(X) = XF$.

<u>Proof</u> Existence is easy: just use coordinates! Its well defined by property (d). More precisely ~~if $\omega = \omega_I\, dx^I$ in~~ pick coordinates $(U, (x^i))$ ~~(Einstein summing)~~, then define

$$d\omega := \varphi^* d((\varphi^{-1})^* \omega).$$

on U. This stitches together to

a smooth $(k+1)$-form because if (V, ψ) another chart, then 14.23(d) implies

$$(\varphi \circ \psi^{-1})^* d((\varphi^{-1})^* \omega) = d((\varphi \circ \psi^{-1})^* (\varphi^{-1})^* \omega)$$

$$= d((\psi^{-1})^* \omega)$$

$$\Rightarrow \quad \varphi^* d((\varphi^{-1})^* \omega) = \psi^* d((\psi^{-1})^* \omega) \text{ on } U \cap V.$$

Uniqueness:

Claim: if $\omega_1|_U = \omega_2|_U \Rightarrow d\omega_1|_U = d\omega_2|_U$.

Let $p \in U$ and pick bump function ψ w/ $\operatorname{supp} \psi \subseteq U$ and $\psi(q) = 1$ on a nbhd of p in U. Then $\psi(\omega_1 - \omega_2) = 0$

$$\Rightarrow 0 = d(\psi(\omega_1 - \omega_2)) = d\psi \wedge (\omega_1 - \omega_2) + \psi \, d(\omega_1 - \omega_2).$$

$$\Rightarrow 0 + \psi(p) \, d(\omega_1 - \omega_2)_p = 0$$

$$\Rightarrow d(\omega_1)_p = d(\omega_2)_p. \quad \checkmark$$

~~In other words,~~

Using bump functions, can see that forms supported on coordinate charts generate $\Omega^*(M)$. Now use (i)–(iv) to finish. $\square$

Prop 14.26

$$F^*(d\omega) = d(F^*\omega)$$

Proof Check in coordinates $(U, \varphi), (V, \psi)$. $\square$

Closed, exact. De Rham.