Prop 14.26

$$F^*(\partial w) = d(F^*_w)$$

Proof Check in coordinates $(\theta, \ell), (V, \tau)$. □

Closed, Exact. De Rham.

Lecture 12.3

Reading: first two sections of Ch. 15

Practice: 15.4, 15.8, 15.10, 15.12, 15.14, 15.13, 15.16

Homework: 15-3, 15-12

Orientations of Vector Spaces

For abstract vector spaces, no preferred orientation... however there is a well-defined way to say whether or not two orientations agree! I.e, well-defined notion of "same orientation."

Def Let $(E_1, \dots, E_n)$ and $(E'_1, \dots, E'_n)$ be two bases of the n-dim'l vector space V. Let $T: V \to V$ $E_i \mapsto E'_i$ be the change of basis isomorphism $T \in GL(V)$. We say $(E_1, \dots, E_n)$ and $(E'_1, \dots, E'_n)$

have same orientation if $\det T > 0$. Otherwise, we say they have opposite orientation.

Easy to check: "same orientation" is an equivalence relation on the set of ordered bases of V, with two equivalence classes.

Def An orientation of V is a choice of one of these two classes. A basis of V agrees with an orientation of V if it is in that equivalence class.

Examples Standard orientations on $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$.

Def If $\dim V = 0$, an orientation of V is a choice of $\pm 1 \in \mathbb{R}$.

Prop 15.3 Let $\dim V = n$, and $\omega \in \Lambda^n(V^*)$ with $\omega \neq 0$. Define

$$O_\omega := \{(E_1, \dots, E_n) \mid \text{ordered bases of } V \text{ s.t. } \omega(E_1, \dots, E_n) > 0\}.$$

Then O_ω is an orientation of V.

Orientations of Manifolds

Def a pointwise orientation of M^n is a choice of orientation of T_pM for each $p \in M^n$. Given a pointwise orientation of M, a local frame (E_i) of TM respects the pointwise orientation if $(E_1|_p, \dots, E_n|_p)$ agrees with the given orientation of T_pM. A pointwise orientation of M is continuous if for every $p \in M$, the given orientation of T_pM can be extended to a continuum! local frame that respects the pointwise orientation. An orientation of M is a choice of continuous pointwise orientation of M. M is orientable if such exists; otherwise M is non-orientable. e.g. Möbius strip non-orientable. $\mathbb{R}^n$ orientable. Open submanifolds

Phew! Fortunately, several equivalent formulations exist. ~~We'll content ourselves w/ sufficient conditions.~~

Note Connected manifold either has 0 or 2 orientations.

Proposition The following are equivalent:

(i) M is orientable

(ii) $\Lambda^n T^*M$ is trivial, where $\dim M = n$

(iii) M admits a consistently oriented coordinate atlas, i.e. an atlas where all transition functions have Jacobians with positive determinant.

Proof

(i) $\Rightarrow$ (ii): Fix an orientation of M. Then we will build a nowhere vanishing n-form on M.

Let $(\Lambda^n_+ T^*M)_p = \{ \omega \in (\Lambda^n T^*M)_p \mid \sigma_v$ agrees with orientation of $T_pM \}$

Prop 15.3.

$(\Lambda^n_+ T^*M)_p$ is a convex subset of $(\Lambda^n T^*M)_p$. Using a partition of unity, can build a smooth section $\uparrow$ with values in $\Lambda^n_+ T^*M$. (See Problem 13-2.)

(ii) $\Rightarrow$ (i) Given non-vanishing $\omega \in \Omega^n(M)$, clearly get a pointwise orientation. Easy to show it's continuous.

(i) $\Rightarrow$ (iii): use the orientation to replace any given atlas w/ an oriented one by replacing any chart $(U_\alpha, (x^i))$ in the atlas whose coordinate frame $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$ doesn't agree with given orientation by a new chart with

$$\tilde{x}^1 := -x^1, \quad \tilde{x}^i := x^i \text{ for } i = 2, \dots, n.$$

Now check its consistently oriented.

(iii) $\Rightarrow$ (i): exercise. Def Local diffeo is orientation preserving. □

Prop 15.15 If $F: M \to N$ a local diffeomorphism and N oriented, then $\exists!$ orientation on M that makes F orientation preserving. Called pullback orientation.

Proof Let ω be orientation form of N and use $F^*\omega$. □

Prop 15.21 If M^n oriented, and $S^{n-1} \subseteq M$ immersed, and N a vector field on M that is nowhere tangent to S, then S is orientable.

Proof Use $i_S^*(N \lrcorner \omega)$ for ω an orientation form of M. □

Ex $\mathbb{RP}^2$ does not injectively immerse in $\mathbb{R}^3$, because $\mathbb{RP}^2$ not orientable.

Prop 15.24 If M is orientable and has boundary, then ∂M is orientable.

Proof Let N be a vector field on M that points outward on ∂N. Use previous proposition. □

Def Induced orientation (or Stokes orientation) on ∂M comes from "outward normal first"

E.g $S^n = \partial B^{n+1}$ orientable.

Q: Is standard orientation on $\mathbb{R}^{n-1} = \partial \mathbb{H}^n$ the Stokes orientation?

$$\left[-\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}}\right] = (-1)^n \left[\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right]$$