Lecture 13.1

Reading: §1–2 of Ch. 16
Practice: 16.7
Homework: 16-4, 16-5, 16-6 (Hairy ball theorem)

So far: integration of covector fields (1-forms) along curves (1-manifolds)

Now: integration of k-forms along k-manifolds. Need orientations.

We will more-or-less mimic exactly what we did for 1-forms:

1. Show that there is a diffeomorphism invariant way to integrate n-forms on open subsets of $\mathbb{R}^n$. Need to keep track of orientation...
2. Use this to integrate on domains of charts on general n-manifolds
3. Use partitions of unity to integrate over all of M.

To do things sufficiently generally, we'll need to fuss about compactness.

Def A domain of integration in $\mathbb{R}^n$ is a bounded subset whose boundary has measure zero.

Def Let $D \subseteq \mathbb{R}^n$ be a domain of integration and $\omega = F\, dx^1 \wedge \cdots \wedge dx^n$ a (continuous) ~~n~~-form on $\bar{D}$.

$$\int_D \omega \overset{\text{def}}{=} \underbrace{\int_D F\, dV}_{\text{usual Riemann integral!}} = \int_D F\, dx^1 \cdots dx^n$$

If $U \subseteq \mathbb{R}^n$ open and ω compactly supported on U, let

$$\int_U \omega \overset{\text{def}}{=} \int_D \omega$$

where D any integration domain w/ $U \subseteq D$ and ω extended by 0 outside $\operatorname{supp} \omega$.

Prop 16.1 $D, E \subseteq \mathbb{R}^n$ open domains of integration $G: \bar{D} \to \bar{E}$ smooth such that $G|_D$ a diffeo to E. If ω an n-form on $\bar{E}$, then

$$\int_D G^* \omega = \pm \int_E \omega$$

with $\pm$ according to whether G orientation preserving or reversing.

Thus, our def'n of integral of an n-form is diffeo invariant (in oriented sense).

Proof: Let (x^i) be std coordinates on ~~E~~ D, (y^i) std coordinates on E, and write $\omega = F\, dy^1 \wedge \cdots \wedge dy^n$.

$$\int_D G^*\omega = \int_D F \circ G \; d(y^1 \circ G) \wedge \cdots \wedge d(y^n \circ G)$$

$$= \int_D F \circ G \; \det DG \; dx^1 \wedge \cdots \wedge dx^n \qquad \text{(Prop. 14.15)}$$

$$= \int_D F \circ G \; \det DG \; dV = \pm \int_D F \circ G \, |\det DG| \, dV$$

$$= \pm \int_E F \, dV \qquad \text{(how Riemann integrals work under reparametrization)}$$

$$= \pm \int_E \omega \qquad \square$$

We expect/want to integrate (at least, compactly supported) n-forms on unbounded open domains; previous lemma not enough. Need a technical lemma:

Lemma 16.2 If $U \subseteq \mathbb{R}^n$ open and $K \subseteq U$ compact, then there exists an open domain of integration D with $K \subseteq D \subseteq U$.

Simple proof that I will skip for time.
Combining two previous results, get

Prop 16.3 Suppose $U, V \subseteq \mathbb{R}^n$ open and $G: U \to V$ a diffeomorphism. If ω is a compactly supported n-form on U, then

$$\int_V \omega = \pm \int_U G^* \omega$$

according to whether G orientation preserving or reversing. $\square$

Compactly supported

Def Let M be oriented n-manifold, $\omega \in \Omega^n(M)$. Let (U, φ) be a chart on M. Assume that $\operatorname{supp} \omega \subseteq U$. Then define

$$\int_M \omega \overset{\text{def}}{=} \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

with a ± 1 according to whether (U, φ) is orientation respecting or not.

Prop 16.4 This def'n doesn't depend on choice of (U, φ) such that $\operatorname{supp} \omega \subseteq U$.

Proof Let $(\tilde U, \tilde\varphi)$ be another chart with $\operatorname{supp}\omega \subseteq \tilde U$. Then $\operatorname{supp}\omega \subseteq U \cap \tilde U$ and

$$\int_{\tilde\varphi(\tilde U)} (\tilde\varphi^{-1})^*\omega = \int_{\tilde\varphi(U\cap\tilde U)} (\tilde\varphi^{-1})^*\omega \overset{16.3}{=} \int_{\varphi(U\cap\tilde U)} (\tilde\varphi\circ\varphi^{-1})^*(\tilde\varphi^{-1})^*\omega$$

think about cases w.r.t. orientations of $\varphi, \tilde\varphi$ ...

$$= \int_{\varphi(U\cap\tilde U)} (\varphi^{-1})^*\omega = \int_{\varphi(U)} (\varphi^{-1})^*\omega.$$

□

Finally

Def Let M be an oriented n-manifold and $\omega \in \Omega^n(M)$ compactly supported. Let $\{U_i\}$ be open cover of $\operatorname{supp}\omega$, with $\{\psi_i\}$ a subordinate smooth partition of unity. Then

by domains of smooth charts (U_i, φ_i)

$$\int_M \omega \overset{\text{def}}{=} \sum_i \int_M \psi_i\omega = \sum_i \left(\pm \int_{\varphi_i(U)} (\varphi_i^{-1})^*(\psi_i\omega) \right)$$

Prop 16.5 This defn is independent of $\{U_i\}$ and $\{\psi_i\}$.

Proof Given two open covers $\{U_i\}$, $\{\tilde U_j\}$ and partitions of unity $\{\psi_i\}$, $\{\tilde\psi_j\}$, get a "refined" partition of unity $\{\psi_i\tilde\psi_j\}$ w.r.t. $\{U_i \cap \tilde U_j\}$

We'll show $\sum_i \int_M \psi_i \omega = \sum_{i,j} \int_M \psi_i \tilde{\psi}_j \omega$.

A similar argument will show $\sum_j \int_M \tilde{\psi}_j \omega = \sum_{i,j} \psi_i \tilde{\psi}_j \omega$

go compare to def'n!

$$\sum_{i,j} \int_M \psi_i \tilde{\psi}_j \omega \overset{\downarrow}{=} \sum_i \int_M \left(\sum_j \psi_i \tilde{\psi}_j \omega \right) \quad \text{(finite sums!)}$$

$$= \sum_i \int_M \psi_i \omega \qquad \text{(def'n of } \{\tilde{\psi}_j\} \text{ being a partition of unity)}$$

$\square$

<u>Note</u>: partitions of unity great for theory, but not so much practice. Do look at Prop 16.8 for a more useful way to actually compute!

<u>Prop 16.6</u> Let $\omega, \eta \in \Omega(M)$, $a, b \in \mathbb{R}$, M, N oriented n-manifolds

(a)

$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$$

(b) If $-M$ denotes M with opposite orientation, then

$$\int_{-M} \omega = -\int_M \omega.$$