Lecture 13.2

Reading: §3 of Ch. 16

Practice: n/a

Last time: integrating (compactly supported...) n-forms on oriented n-manifolds.

Of course, if $\omega \in \Omega^k(M)$ (compactly supported) and $K \subseteq M$ an immersed, oriented k-dim'l submanifold (for which $i_K^* \omega$ compactly supported) then we can define

$$\int_K \omega := \int_K i_K^* \omega.$$

(Note: no "compatibility" of orientation b/w K and M needed here.)

In particular, if M oriented and has boundary, we saw last week that there is a standard way using "outward normal first" convention. Thus, we can unambiguously write, for any $(n-1)$-form ω

$$\int_{\partial M} \omega \overset{\text{def}}{=} \int_{\partial M} i_{\partial M}^* \omega$$

where ∂M has this induced orientation.

Stokes's theorem Let $\omega \in \Omega^{n-1}(M)$ be compactly supported. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

We'll prove this today. First, some special cases worth making explicit.

<u>Corollary</u> If $\partial M = \phi$, then

$$\int_M d\omega = 0.$$

i.e., exact n-forms have vanishing integrals on (oriented) closed n-manifolds

<u>Corollary</u> If $d\omega = 0$, then

$$\int_{\partial M} \omega = 0.$$

i.e., closed $n-1$ forms have vanishing boundary integrals.

<u>Corollary</u> If $K \subseteq M$ closed, compact, oriented k-dim'l submanifold and ω a closed k-form on M with $\int_K \omega \neq 0$, then:

(a) ω is not exact on M and
(b) K is not the boundary of an orientable compact embedded submanifold in M.

Ex $\omega = \frac{x}{x^2+y^2} dy - \frac{y}{x^2+y^2} dx$ has nonzero integral over S^1. Thus S^1 is not boundary of any oriented compact 2-mfd in $\mathbb{R}^2 - \{0\}$. Cool!!!

Proof of Stokes' Theorem

By def'n, integration on M or ∂M involves a partition of unity. Since M has boundary, there are two types of charts. Thus, even the special case that $\omega \in \Omega^{n-1}(M)$ is compactly supported in a single chart of M really has two subcases. These will constitute most of the work.

For now, let's assume the theorem holds for all (compactly supported) $n-1$ forms with support in a single chart. Consider an arbitrary $n-1$ form $\omega \in \Omega^{n-1}(M)$ of compact support:

$$\cancel{\int_M d\omega = \sum_i \int}$$

$$\int_{\partial M} \omega = \sum_i \int_{\partial M} \psi_i \omega \overset{\text{by assumption}}{=} \sum_i \int_M d(\psi_i \omega)$$

Recall: d a (graded) derivation of $\Omega^*(M)$

$$= \sum_i \int_M d\psi_i \wedge \omega + \psi_i \, d\omega$$

$$= \int_M d\Big(\sum_i \psi_i\Big) \wedge \omega + \int_M \Big(\sum \psi_i\Big) d\omega$$

$$= \int_M d(1) \wedge \omega + \int_M 1 \, d\omega = \int_M d\omega.$$

Okay, so we need to show the theorem is true for $\omega \in \Omega^{n-1}(M)$ compactly supported in a single chart of M.

Before doing this, we do two extra special cases: (a) $M = \mathbb{H}^n$ ~~& ω~~
(b) $M = \mathbb{R}^n$, ~~ω ... ω compactly supported~~

Case (a) Recall

$$\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}.$$

B/c ω compactly supported, exists $R \in \mathbb{R}$ w/ $\operatorname{supp} \omega \subseteq A := [-R, R] \times \cdots \times [-R, R] \times [0, R]$. In std coordinates, have

$$\omega = \sum_{i=1}^{n} \omega_i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

$$\Rightarrow d\omega = \cdots = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \quad \text{(no hat!)}$$

$$\Rightarrow \int_{\mathbb{H}^n} d\omega = \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) \, dx^1 \cdots dx^n$$

By Fubini theorem + FToC + compact support, for any $1 \leq i \leq n-1$, have

$$\int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial w_i}{\partial x^i}(x)\, dx^1 \cdots dx^n$$

$$= \int_0^R \underbrace{\int_{-R}^R \cdots \int_{-R}^R}_{n-1} \frac{\partial w_i}{\partial x^i}(x)\, dx^i\, dx^1 \cdots \widehat{dx^i} \cdots dx^n$$

$$= \int_0^R \underbrace{\int_{-R}^R \cdots \int_{-R}^R}_{n-1} \Big[w_i(x) \Big]_{x^i=-R}^{x^i=R} dx^1 \cdots \widehat{dx^i} \cdots dx^n$$

$= 0$. Plugging this in and applying FToC + Fubini again:

$$\int_{\mathbb{H}} d\omega = (-1)^{n-1} \underbrace{\int_{-R}^R \cdots \int_{-R}^R}_{n-1} \int_0^R \frac{\partial w_n}{\partial x^n}(x)\, dx^n\, dx^1 \cdots dx^{n-1}$$

$$= (-1)^{n-1} \int_{-R}^R \cdots \int_{-R}^R \Big[w_n(x^1, \dots, x^{n-1}, R) - w_n(x^1, \dots, x^{n-1}, 0) \Big] dx^1 \cdots dx^{n-1}$$

$$= (-1)^n \int_{-R}^R \cdots \int_{-R}^R w_n(x^1, \dots, x^{n-1}, 0)\, dx^1 \cdots dx^{n-1}.$$

On the other hand,

$$\int_{\partial \mathbb{H}^n} \omega = \sum_i \int_{\partial \mathbb{H}^n \cap A} w_i(x^1, \dots, x^{n-1}, 0)\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

But $dx^n|_{\partial \mathbb{H}^n} = 0$! Thus

$$\int_{\partial \mathbb{H}^n} \omega = \int_{A \cap \partial \mathbb{H}^n} \omega_n (x^1, \dots, x^{n-1}, 0)\, dx^1 \wedge \dots \wedge dx^{n-1}$$

$$= \int_{-R}^{R} \cdots \int_{-R}^{R} \omega_n (x^1, \dots, x^{n-1}, 0)\, dx^1 \cdots dx^{n-1}$$

boundary orientation inherited from $\mathbb{H}^n$, which agrees w/ std orientation on $\mathbb{R}^{n-1} (\cong \partial \mathbb{H}^n)$ iff n is even. Thus, Stokes' theorem holds for $M = \mathbb{H}^n$.

Case (b) If $M = \mathbb{R}^n$, the exact same argument goes through, except it's even easier because the boundary term doesn't exist, in particular

$$\int_{\mathbb{R}^n} d\omega = 0 = \int_{\partial \mathbb{R}^n} \omega .$$

because $\partial \mathbb{R}^n = \emptyset$.

Finally, can use the above two cases to prove Stokes's theorem in the final remaining case: M arbitrary, but ω a compactly supported $(n-1)$-form with supp ω inside a single chart. $\square$