Lecture 13.3

Reading: §1-2 of Ch. 17
Practice: n/a
Homework: 17-1, ~~17-8, 17-13~~

Today: construction and elementary properties of de Rham cohomology

Definition

Def A p-form $\omega \in \Omega^p(M)$ is *closed* if $d\omega = 0$, and *exact* if exists $\eta \in \Omega^{p-1}(M)$ such that $d\eta = \omega$.

Note: $d^2 = 0 \Rightarrow$ "exact $\Rightarrow$ closed"

Question: closed $\overset{?}{\Rightarrow}$ exact?

No! De Rham cohomology ~~measures~~ quantifies, for a given manifold, the failure for closed forms to be exact.

Def Let

$$Z^p(M) := \operatorname{Ker}(d: \Omega^p(M) \to \Omega^{p+1}(M))$$
$$= \{\text{closed } p\text{-forms}\} \leq \Omega^p(M).$$
$$B^p(M) := \operatorname{Im}(d: \Omega^{p-1}(M) \to \Omega^p(M))$$
$$= \{\text{exact } p\text{-forms}\} \leq Z^p(M) \leq \Omega^p(M).$$

$$
\begin{tikzcd}
\Omega^{p-1}(M) \arrow[r, "d"] \arrow[d, phantom, sloped, "\supseteq"] \arrow[ddr] & \Omega^{p}(M) \arrow[r, "d"] \arrow[d, phantom, sloped, "\supseteq"] \arrow[ddr] & \Omega^{p+1}(M) \arrow[d, phantom, sloped, "\supseteq"] \\
Z^{p-1}(M) \arrow[d, phantom, sloped, "\supseteq"] \arrow[ddr] & Z^{p}(M) \arrow[d, phantom, sloped, "\supseteq"] \arrow[ddr] & Z^{p+1}(M) \arrow[d, phantom, sloped, "\supseteq"] \\
\mathcal{B}^{p-1}(M) \arrow[d, phantom, sloped, "\supseteq"] \arrow[dr] & \mathcal{B}^{p}(M) \arrow[d, phantom, sloped, "\supseteq"] \arrow[dr] & \mathcal{B}^{p+1}(M) \arrow[d, phantom, sloped, "\supseteq"] \arrow[dr] & {} \\
\{0\} & \{0\} & \{0\} & \cdots
\end{tikzcd}
$$

The <u>de Rham cohomology of M in degree p</u> is the vector space

$$H^p_{dR}(M) := Z^p(M)/\mathcal{B}^p(M).$$

<u>Note</u>: $H^p_{dR}(M) = 0 \iff$ every closed p-form is exact.

<u>Elementary examples</u>

<u>Notation</u>: if $\omega \in Z^p(M)$, we let $[\omega] \in H^p_{dR}(M)$ denote its image in $H^p_{dR}(M)$. Note that $[\omega] = [\omega']$ iff $\omega - \omega'$ is exact, in which case we say ω and ω' are <u>cohomologous</u>.

1. $H^1_{dR}(\mathbb{R}^2 - \{0\}) \neq 0$ b/c $\dfrac{x\,dy - y\,dx}{x^2+y^2}$ not conservative.

2. $\Omega^p(M_1 \sqcup M_2) \cong \Omega^p(M_1) \oplus \Omega^p(M_2)$

$$\begin{array}{ccc} \Omega^p(M_1 \sqcup M_2) & \cong & \Omega^p(M_1) \oplus \Omega^p(M_2) \\ \cup\!\! \mid & & \cup\!\! \mid \\ Z^p(M_1 \sqcup M_2) & \cong & Z^p(M_1) \oplus Z^p(M_2) \\ \cup\!\! \mid & & \cup\!\! \mid \\ B^p(M_1 \sqcup M_2) & \cong & B^p(M_1) \oplus B^p(M_2) \end{array}$$

$$\Rightarrow H^p_{dR}(M_1 \sqcup M_2) \cong H^p_{dR}(M_1) \oplus H^p_{dR}(M_2).$$

3. If M connected ~~other~~ and $F \in \Omega^0(M) = C^\infty(M)$ closed $\Leftrightarrow dF = 0 \Leftrightarrow F$ constant. Thus $Z^0(M) = \mathbb{R}$. On the other hand, $B^0(M) = \{0\}$. Thus

$$H^0_{dR}(M) = \mathbb{R}.$$

Functoriality

Prop 17.2

If $F: M \to N$ smooth ~~then~~ and $[\omega] \in H^p_{dR}(N)$, then get a well-defined ~~homomorphism~~ linear map

$$F^*: H^p_{dR}(N) \to H^p_{dR}(M)$$
$$[\omega] \mapsto [F^*\omega],$$

called the induced map on de Rham cohomology.

Has following properties:

(a) $Id^* = Id$

(b) $(G \circ F)^* = F^* \circ G^*$

} Contravariant functor!

Proof $F^*(d\omega) = d(F^*\omega)$.

$\Rightarrow$ F^*(closed) is closed $\wedge$
F^*(exact) is exact.

$$\begin{array}{ccccc} B^p(N) & \leq & Z^p(N) & \leq & \Omega^p(N) \\ \downarrow F^* & & \downarrow F^* & & \downarrow F^* \\ B^p(M) & \leq & Z^p(M) & \leq & \Omega^p(M) \end{array}$$

$\Rightarrow$ $F^*: Z^p(N)/B^p(N) \to Z^p(M)/B^p(M)$
well-defined. (a) and (b) are easy. $\square$

Corollary If M and N are diffeomorphic, then $H^p_{dR}(M) \cong H^p_{dR}(N)$ $\forall p$. $\square$

In fact, $H^p_{dR}(-)$ is a much weaker invariant, depending only on homotopy type of M.

Def $F, G: M \xrightarrow{cts} N$ are homotopic ($F \simeq G$) if $\exists$ $H: M \times [0,1] \xrightarrow{cts} N$ w/ $H|_{M\times\{0\}} = F$, $H|_{M\times\{1\}} = G$.

Homotopy invariance

Recall:

Def $F: M \xrightarrow{cts} N$ is a *homotopy equivalence* if there exists $G: N \to M$ such that

$$F \circ G \simeq \mathrm{Id}_N \quad \text{and}$$
$$G \circ F \simeq \mathrm{Id}_M$$

Proposition 17.10 If M, N smooth manifolds and $F, G : M \to N$ homotopic smooth maps, then for every p, $F^*, G^* : H^p_{dR}(N) \to H^p_{dR}(M)$ are the same map.

Proof is rather involved... need to build a "homotopy operator" b/w F^* and G^*. Basic idea not too deep, but Lee uses Cartan's magic formula to check the "obvious" thing does the trick---

Theorem 17.11 If M and N homotopy equivalent, then $H^p_{dR}(M) \cong H^p_{dR}(N)$ $\forall p$.

Proof Let $F : M \to N$ be a homotopy equivalence with homotopy inverse $G : N \to M$. By Whitney approximation theorem, there exists smooth maps $\tilde{F} : M \to N$ and $\tilde{G} : N \to M$ homotopic to F, G. Then $\tilde{F}$ a homotopy equivalence w/ homotopy inverse $\tilde{G}$. (Easy to check.)

~~Then $(\tilde{F} \circ \tilde{G})^* = \tilde{G}^* \circ \tilde{F}^*$~~

Then

$$\tilde{F}^* \circ \tilde{G}^* = (\tilde{G} \circ \tilde{F})^* \overset{17.10}{=} (\mathrm{Id})^* = \mathrm{Id}.$$

Similarly $\tilde{G}^* \circ \tilde{F}^* = \mathrm{Id}$. Thus, $(\tilde{F}^*)^{-1} = \tilde{G}^*$ and $(\tilde{G}^*)^{-1} = \tilde{F}^*$, so $\tilde{F}^*$ an isomorphism. $\square$

__Aside__: We can combine Prop 17.10 w/ Whitney approximation to get induced maps $F^* : H^p_{dR}(N) \to H^p_{dR}(M)$ for any continuous (!) maps $F : M \to N$.

Corollary If M and N smooth manifolds that are homeomorphic, then $H^p_{dR}(M) \cong H^p_{dR}(N)$ for all p. $\square$

Note: diffeomorphic $\Rightarrow$ homeomorphic $\Rightarrow$ homotopy equivalent

$\not\Leftarrow$ exotic spheres $\qquad \not\Leftarrow$ lens spaces

~~Theorem 17.13~~

Def A topological space is contractible if its identity map is homotopic to a constant map.

Example "Star-shaped" open subsets of $\mathbb{R}^n$ such as $\mathbb{R}^n$, $\mathbb{B}^n$, $\mathbb{H}^n$

Theorem 17.13 If M is contractible, then $H^p_{dR}(M) = 0$ for $p \geq 1$.

Proof Since $Id_M \simeq c$, a constant map for some $c \in M$, we see $Id = Id^*: H^p_{dR}(M) \to H^p_{dR}(M)$ is 0 map. Then, $H^p_{dR}(M) = 0$. (Why $p > 0$?!) $\square$

Corollary (Generalized Poincaré lemma)
Every closed form on a star-shaped domain is exact. Ex S^1 is not contractible.