Lecture 14.1

Reading: §3 of Ch. 17, thru Exercise 17.29.
Practice: 17.22, 17.24, 17.29
Homework: ~~17-2~~, 17-8, ~~17-13~~.

Today: describe a tool for computing de Rham cohomology via "cut and paste." Called Mayer-Vietoris sequence, use it in several applications.

Def A sequence of vector spaces and linear maps

$$\cdots \to V^{p-1} \xrightarrow{F_{p-1}} V^{p} \xrightarrow{F_p} V^{p+1} \to \cdots$$

is *exact* if for each p

$$\operatorname{Im} F_{p-1} = \operatorname{Ker} F_p.$$

Thm 17.20 (Mayer-Vietoris) Let M be a smooth manifold, $U, V \subseteq M$ open with $M = U \cup V$. Then there exists maps δ making an exact sequence

$$\cdots \to H^{p-1}_{dR}(U\cap V) \xrightarrow{\delta} H^p_{dR}(M) \xrightarrow{k^* \oplus l^*} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{i^* - j^*} H^p_{dR}(U\cap V) \xrightarrow{\delta} \cdots$$

where

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U\cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & M \\ & V \arrow[ur, "l"'] & \end{tikzcd}$$

inclusions.

$\boxed{1/6}$

We are going to prove this. In particular we're not going to explain ~~what~~ how δ defined. Nevertheless, this theorem is extremeley useful!

<u>Thm</u> For $n \geq 1$,

$$H^p_{dR}(S^n) = \begin{cases} \cancel{\mathbb{R}, \text{ for } 0} \\ \mathbb{R}, & \text{if } p=0 \text{ or } p=n \\ 0, & \text{else.} \end{cases}$$

<u>Proof</u> S^n connected $(n \geq 1) \Longrightarrow H^0_{dR}(S^n) \cong \mathbb{R}$ ✓
~~Otherwise, w~~ For $p \geq 1$, we'll use induction on n + Mayer-Vietoris.

~~Let~~ For $n=1$, we know $H^1_{dR}(S^1)$ has dimension <u>at least</u> 1. In fact, Thm 17.17 $\Rightarrow H^1_{dR}(S^1) \cong \mathbb{R}$.

Suppose the theorem has been proved for S^{n-1}. Let $U, V \subseteq S^n$ be domains of two stereographic charts. Then

$$U \cap V \cong \mathbb{R}^n - \{0\} \simeq S^{n-1}.$$ Apply M-V:

$$H^{p-1}(U) \oplus H^{p-1}(V) \to H^{p-1}(S^{n-1}) \xrightarrow{\delta} H^p(S^n) \to \underset{\substack{\cong \\ H^p(\mathbb{R}^n) \\ 0}}{H^p(U)} \oplus \underset{\substack{\cong \\ H^p(\mathbb{R}^n) \\ 0}}{H^p(V)} \to H^p(S^{n-1})$$

By exactness, if $p \geq 2$, d is an isomorphism

$$0 \oplus 0 \to H^{p-1}(S^{n-1}) \underset{\cong}{\xrightarrow{d}} H^p(S^n) \to 0 \oplus 0$$

✓

If $p = 1$:

$$\underset{\mathbb{R}}{\underset{\cong}{H^0(\mathbb{R}^n)}} \oplus \underset{\mathbb{R}}{\underset{\cong}{H^0(\mathbb{R}^n)}} \xrightarrow{i^* - j^*} H^0(S^{n-1}) \xrightarrow{d} H^1(S^n) \to \underset{0}{H^1(\mathbb{R}^n)} \oplus \underset{0}{H^1(\mathbb{R}^n)} \to \cdots$$

surjective (why?) $\Rightarrow$ 0 map $\Rightarrow$ injective + $\nearrow$

$\Downarrow$

$H^1(S^n) = 0$. $\square$

<u>Note</u>: If ω an orientation form on S^n, then $[\omega] \neq 0 \in H^n(S^n)$; thus $[\omega]$ <u>spans</u> $H^n(S^n) \cong \mathbb{R}$.

<u>Why?</u> $\int_{S^n} \omega > 0 \overset{\text{Stokes'}}{\Rightarrow} \omega$ not exact

~~Corollary~~ In fact, we can generalize this line of reasoning to <u>any</u> ~~non~~ orientable manifold. ~~Compact case easy, non-compact a little harder, and we'll discuss~~ But this step requires some care w/ noncompactness. More details on Wednesday. For now, two corollaries:

Corollary 17.23 If $n \geq 2$ and $x \in \mathbb{R}^n$, then

$$H^p(\mathbb{R}^n - \{x\}) = \begin{cases} \mathbb{R}, & p = 0 \text{ or } p = n-1 \\ 0, & \text{else.} \end{cases}$$

a closed $n-1$ form η on $\mathbb{R}^n - \{x\}$ is exact iff $\int_S \eta = 0$ for some (hence every) sphere $S \subseteq \mathbb{R}^n$ centered at x.

Corollary 17.25 If $n \geq 2$, ~~x ∈ U~~ $U \subseteq \mathbb{R}^n$ open, $x \in U$, then ~~$H^{n-1}(U) \neq 0$~~

$$H^{n-1}(U - \{x\}) \neq 0.$$

Big theorem! { Thm 17.26 The dimension of a topological manifold is a homeomorphism invariant.

Proof Let M^m, N^n be topological manifolds. If m or $n = 0$, easy. So assume $m > n \geq 1$, let ~~Let $U \subseteq M$ be open, homeomorphic to $\mathbb{R}$~~ $F: M \to N$ a homeomorphism.

Let $x \in M$ with an open nbhd $U \ni x$ homeomorphic to $\mathbb{R}^m$. We can use this homeo to make U a smooth m-manifold, and therefore define de Rham cohomology on U. ~~In particular,~~ But de Rham cohomology is a homeomorphism invariant, hence, in particular

$F: M^m \xrightarrow{\cong} N^n$

Pick $x \in N$.

Can find $V \subseteq M^m$ and $x \in U \subseteq N^n$ with $F^{-1}(U) \subseteq V$, $U \cong \mathbb{R}^n$, $V \cong \mathbb{R}^m$. ~~Since $m \geq n$,~~

~~$H^{m-1}_{dR}(U - \{x\}) \cong H^{m-1}_{dR}(\mathbb{R}^n - \{0\})$~~

Use to make U a smooth n-mfd $\cong \mathbb{R}^n$

Use to make U a smooth m-mfd. $\subseteq \mathbb{R}^m (\cong V)$

On one hand, b/c $m > n$ and $U - \{x\} \cong \mathbb{R}^n - \{0\}$

$$H^{m-1}_{dR}(U - \{x\}) \cong H^{m-1}_{dR}(\mathbb{R}^n - \{0\}) \underset{17.23}{=} 0$$

On other hand, b/c U homeomorphic to an open subset of $\mathbb{R}^m$

$$H^{m-1}_{dR}(U - \{x\}) \underset{17.25}{\neq} 0$$

$\square$