Lecture 14-2

Reading: finish §3 of Ch. 17, §4.
Practice: 17.37
Homework: 17-13

Today: compute "top degree" cohomology of all manifolds, in both compact and non-compact settings. Using this, can associate an integer to any continuous map $F: M \to N$ between compact oriented n-manifolds, called the degree.

Compactly supported cohomology

Aside: H^p_c is not functorial in general: only for proper maps

Def $\Omega^p_c(M) := \{$compactly supported p-forms$\}$

$\subseteq$ ← equal if M compact

$\Omega^p(M)$.

Note that if $\omega \in \Omega^p_c(M)$, then $d\omega \in \Omega^{p+1}_c(M)$. So may define the compactly supported cohomology

$$H^p_c(M) := \frac{\ker\left(d: \Omega^p_c(M) \to \Omega^{p+1}_c(M)\right)}{\operatorname{Im}\left(d: \Omega^{p-1}_c(M) \to \Omega^p_c(M)\right)}$$

Lemma 17.27 (Poincaré lemma w/ compact support) Let $n \geq p \geq 1$, and $\omega \in \Omega^p_c(\mathbb{R}^n)$. If $p = n$, suppose furthermore that $\int_{\mathbb{R}^n} \omega = 0$. Then there exists a compactly supported $p-1$ form $\eta \in \Omega^{p-1}_c(\mathbb{R}^n)$ with $d\eta = \omega$.

Using partitions of unity, can prove for M orientable and connected:

	Compact M	noncompact M
H^n_{dR}	$\mathbb{R}$	0
H^n_c	$\mathbb{R}$	$\mathbb{R}$

orientable connected n-mfd's top cohomology

Using covering spaces, can prove for M^n non-orientable:

$$\boxed{H^n_{dR}(\text{non-orientable}) = H^n_c(\text{non-orientable}) = 0.}$$

nonorientable n-mfd's top cohomology

In fact, if M connected and oriented, and ω is an ~~orientation~~ compactly supported n-form, can integrate over M:

$$I : \Omega^n_c(M) \longrightarrow \mathbb{R}$$
$$\omega \longmapsto \int_M \omega.$$

Of course, $I(d\eta) = 0$ for a compactly supported $n-1$ form η. Thus I

factors through an map

$$\Omega^n_c(M) \xrightarrow{/B^n_c(M)} H^n_c(M) \xrightarrow{I} \mathbb{R} \qquad (\star)$$

isomorphism!

Degree Theory

Thm 17.35 Suppose M, N compact, connected, oriented, smooth mfds of dimension n. If $F: M \to N$ a smooth, then there exists a unique integer k called the degree of F such that:

(a) for any $\omega \in \Omega^n(N)$

$$\int_M F^* \omega = k \int_N \omega$$

(b) if $q \in N$ a regular value of F, then

$$k = \sum_{x \in F^{-1}(q)} \operatorname{sgn}(x)$$

where $\operatorname{sgn}(x) = \begin{cases} +1 & \text{if } dF_x \text{ orientation preserving} \\ -1 & \text{"} \quad \text{"} \quad \text{reversing.} \end{cases}$

~~Proof of (a)~~ (a) (★) above says that two n-forms on ~~an n-manifold~~ a (connected, oriented) n-manifold have same integral iff cohomologous.

Pick $\theta \in \Omega^n(N)$ such that $\int_N \theta = 1$, and let $k := \int_M F^*\theta$. If $\omega \in \Omega^n(N)$, there exists ~~an integer~~ $a \in \mathbb{R}$ such that ω cohomologous to $a\theta$. Thus,

$$\int_M F^*\omega = \int_M F^*(a\theta) = a\int_M F^*\theta = ak = ak\int_N \theta$$

$$= k\int_N (a\theta) = k\int_N \omega.$$

(b) Idea: since $q \in N$ a regular value, $F^{-1}(q) \subseteq M$ a finite set of points $\{x_1, \dots, x_m\}$.
Can find regular coordinate ball $W \ni q$ ~~regular~~ such that $F^{-1}(W)$ is a disjoint union of regular coordinate balls $V_i \ni x_i$.

Let $\omega \in \Omega^n(N)$ compactly supported in W w/ $\int_N \omega = \int_W \omega = 1$. By (a), $\int_M F^*\omega = k$.

By construction

$$\int_M F^*\omega = \sum \int_{V_i} F^*\omega$$

But $F|_{V_i}$ is a diffeomorphism

$F|_{V_i} : V_i \xrightarrow{\cong} W$ so

$$\int_{V_i} F^*\omega = \pm \int_W \omega = \pm 1 = \begin{cases} +1 & \text{if } F|_{V_i} \text{ orientation preserving} \\ -1 & \text{else.} \end{cases}$$

$\square$

<u>Properties of degree</u>

(a) If $F: M \to N$, $G: N \to P$, then
$\deg(G \circ F) = \deg(G)\deg(F)$

(b) If $F: M \xrightarrow{\cong} N$, then
$\deg(F) = \pm 1$ depending on orientations

(c) If $F_0, F_1 : M \to N$ homotopic, then
$\deg F_0 = \deg F_1$.

<u>Next time</u>: Brouwer fixed point theorem