Lecture 14.3

Reading: §1-2 of Ch. 13
Practice: 13.7, 13.10, 13.13, 13.15, 13.22, 13.23, 13.27
Homework: 16-18

Last time: degree of a smooth map b/w oriented compact n-manifolds

Def: Let M, N be compact oriented n-manifolds and $F: M \to N$ a continuous map. The degree of F, $\deg F$, is the degree of any smooth map homotopic to F (using Whitney approximation).

Thm 17.38 N: compact oriented smooth n-mfd ($\partial N = \emptyset$)
X: compact, oriented smooth $(n+1)$-mfd w/ connected (nonempty?) boundary.
If $F: \partial X \to N$ cts w/ a cts extension to X, then $\deg F = 0$.

Proof By Whitney, might as well assume F smooth w/ a smooth extension $F: X \to N$. Let ω be a smooth n-form on N. Then

$$\int_{\partial X} F^*\omega = \int_{\partial X} F^*\omega \overset{\text{Stokes'}}{=} \int_X d(F^*\omega) = \int_X F^*(d\omega) = 0.$$

$$\Rightarrow \deg F = 0. \quad \square$$

Thm 17.B9 (Brouwer fixed point theorem)
Every cts map $\overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$ has a fixed point.

Proof Suppose $F: \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$ cts w/out any fixed points. Then

$$G: \overline{\mathbb{B}^n} \to \mathbb{S}^{n-1}$$
$$x \mapsto \frac{x - F(x)}{|x - F(x)|}$$

is cts. Let $g = G|_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$.

By previous theorem, $\deg g = 0$.

On other hand,

$$H: \mathbb{S}^{n-1} \times I \to \mathbb{S}^{n-1}$$
$$(x,t) \mapsto \frac{x - tF(x)}{|x - tF(x)|}$$

is a continuous homotopy b/w g and $\mathrm{Id}_{\mathbb{S}^{n-1}}$.
Thus, $\deg g = \deg \mathrm{Id} = 1$.

Contradiction! $\square$

Note: Nash's original proof of existence of equilibria in finite games uses Brouwer fixed point. Earned him a Nobel prize in economics.

~~Riem~~ Riemannian manifolds

Recall: if V a finite dim'l vector space over $\mathbb{R}$, then an inner product is:

$$g : V \times V \to \mathbb{R}$$
$$(v,w) \mapsto g(v,w)$$

that is:

(i) bilinear

(ii) symmetric

(iii) positive definite: $g(v,v) \geq 0$, w/equality iff $v = 0$.

In other words: an inner product on V is a ~~symmetric posi~~ symmetric covariant 2-tensor that is positive definite.

Def Let M be a smooth manifold. A Riemannian metric on M is a smooth, symmetric, covariant 2-tensor field g on M that is positive definite at every point of M. The pair (M, g) is called a Riemannian manifold.

Prop 13.3 Every smooth manifold admits a Riemannian metric.

Ex The Euclidean metric on $\mathbb{R}^n$ is given in standard coordinates (x^i) by

$$g = \delta_{ij} \underbrace{dx^i dx^j}_{\text{symmetric product}} = \delta_{ij}\, dx^i \otimes dx^j$$

If $v, w \in T_p\mathbb{R}^n$, $g(v,w) = g_p(v,w) = \delta_{ij}\, dx^i(v)\, dx^j(w)$

$$= \delta_{ij} v^i w^j = v^i w^i = v \cdot w.$$

Proof Let $\{(U_i, \varphi_i)\}$ be a covering of M by smooth charts. For each i, define $g_i := \varphi_i^* g_{Eucl}$. Let $\{\psi_i\}$ be smooth partition of unity subordinate to $\{U_i\}$. Define

$$g = \sum_i \psi_i g_i .$$

Now easy to check g a Riemannian metric $\square$

Recall that Euclidean metric on $\mathbb{R}^n$ is responsible for "all (Euclidean) geometry in $\mathbb{R}^n$."

Thus, we think of a Riemannian metric as a way to "do geometry" on a smooth manifold.

Smooth manifold: topological structures that allows for a ~~theory~~ generalization of differential and integral calculus in $\mathbb{R}^n$

Riemannian manifold: ~~differential~~ smooth structures that allow for a generalization of analytic geometry in $\mathbb{R}^n$.

Note: we want to think of a Riemannian metric g on a smooth manifold as additional structure/data. There is (in general) no "canonical" metric we can put on a smooth manifold — we must choose one (similar to how a smooth manifold requires a choice of smooth atlas on a topological manifold)

Defs Let (M, g) be a Riemannian mfld.
- length or norm of $v \in T_pM$ is

$$|v|_g = g_p(v,v)^{1/2}.$$

- angle b/w $v, w \in T_pM$ is $\theta \in [0, \pi]$ s.t.

$$\cos\theta = \frac{g(v,w)}{|v|_g\,|w|_g}.$$

- $v, w \in T_pM$ are orthogonal if $g(v,w) = 0$.

In Riemannian geometry, care about properties that are invariant under <u>isometries</u>

Def $F:(M,g) \to (N,h)$ is an <u>isometry</u> if $F: M \to N$ a diffeo and
$$h(F(v), F(w)) = g(v,w)$$
for all $v, w \in T_pM$, $p \in M$. A <u>local isometry</u> is similar, except we only require F be a local isometry.

Def (M,g) is <u>flat</u> if it is locally isometric to $(\mathbb{R}^n, g_{Eucl})$.

The most important obstruction to flatness is <u>curvature</u>. Take MA 661 for more.

Just for fun:

<u>Nash Embedding Theorem</u>

Every Riemannian "n" manifold admits an <u>isometric embedding</u> into $(\mathbb{R}^m, g_{Eucl})$ for
$$m \geq \frac{n(n+1)(3n+11)}{2}.$$

"Geometric" analog of Whitney embedding.