Lecture 15-1

Reading: §3 of Ch. 13, §3.1 of Ch 14, §3 of Ch 15
§5 of Ch. 16

Practice: 14.25, 15.30, 16.31

Homework: 16-18

Tangent - cotangent isomorphism

Basic idea: if V a vector space w/ inner product g, then get a homomorphism

$$\hat{g}: V \to V^*$$
$$v \mapsto g(v, -).$$

This is an injective map if g is positive definite (or more generally, non-degenerate), and therefore an isomorphism if V is moreover finite dimensional.

Generalizing: If (M, g) a Riemannian manifold, we can "conflate" covariant and contravariant vectors. Consider

$$\hat{g}: TM \to T^*M$$
$$v \mapsto g(v, -)$$

Can be shown relatively easily that this is a smooth vector bundle immersion; because $\dim TM = \dim T^*M$, it follows that $\hat{g}: TM \to T^*M$ is a vector bundle isomorphism.

Get maps converting covariant tensor fields to contravariant fields and vice versa. Called index raising and lowering, respectively.

These are inverse operations. For example:

$$\flat : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$$
$$X \mapsto g(X, -) =: X^\flat$$

$$\# : \mathfrak{X}^*(M) \to \mathfrak{X}(M)$$
$$\omega \mapsto \omega^\# := \flat^{-1}(\omega)$$

In coordinates: if $X = X^i \frac{\partial}{\partial x^i}$, then

$X^\flat = g_{ij} X^i dx^j$. Why?

$$(g_{ij} X^i dx^j)(Y^k \tfrac{\partial}{\partial x^k}) = g_{ij} X^i Y^k dx^j \left(\tfrac{\partial}{\partial x^k}\right)$$
$$= g_{ij} X^i Y^j = X^\flat(Y).$$

We will write $X^\flat = X_j \, dx^j$ where

$$X_j := g_{ij} X^i$$ "flattening / lowering the index"

Inversely: let g^{ij} be the matrix inverse of g_{ij}, so $g^{ij} g_{jk} = \delta^i_k$. If $\omega = \omega_j dx^j$, then

$$\omega^\# = \omega^i \frac{\partial}{\partial x^i} \quad \text{where } \omega^i = g^{ij} \omega_j.$$

"sharpening / raising the index."

Gradients

If $F : M \to \mathbb{R}$ smooth and g a metric on M, we define the *gradient* of F to be

$$\operatorname{grad} F := (dF)^\#.$$

Thus, grad F is unique vector field on M

such that

$$\mathrm{grad}: C^\infty(M) \to \mathfrak{X}(M)$$

$$\begin{tikzcd} C^\infty(M) \arrow[r, "\mathrm{grad}"] \arrow[d, equal] & \mathfrak{X}(M) \\ \Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[u, bend right, "\#"'] \arrow[d, equal] \\ & \mathfrak{X}^*(M) \end{tikzcd}$$

$$\cancel{\langle \mathrm{grad}\, F, - \rangle_g}$$

$$g(\mathrm{grad}\, F, -) = dF.$$

<u>In coordinates:</u> ~~if $dF =$~~

$$dF = \frac{\partial F}{\partial x^i}\, dx^i$$

$$\Rightarrow \mathrm{grad}\, F = g^{ij} \frac{\partial F}{\partial x^i} \frac{\partial}{\partial x^j}$$

If $(M, g) = (\mathbb{R}^n, g_{Euc})$, this reduces to the familiar

$$\mathrm{grad}\, F = \delta^{ij} \frac{\partial F}{\partial x^i} \frac{\partial}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial F}{\partial x^i} \frac{\partial}{\partial x^i}.$$

We can also define a divergence operator on any Riemannian manifold. But first, need:

Riemannian Volume Forms

<u>Prop 15.29</u> Suppose (M, g) is an oriented Riemannian manifold (of dim ≥ 1). Then there exists a unique smooth orientation form $\omega_g \in \Omega^n(M)$ called the Riemannian volume form such that

$$\omega_g(E_1, \dots, E_n) = 1$$

for every local oriented orthonormal frame (E_i).

Proof Uniqueness is easy: if $(E_1, \dots, E_n)$ a local oriented orthonormal frame (on U) w/ dual coframe $(\varepsilon^1, \dots, \varepsilon^n)$, then

$$\omega_g = f\, \varepsilon^1 \wedge \dots \wedge \varepsilon^n$$

for some positive $f : U \to \mathbb{R}$. In fact, $f = 1$.
Thus, $\omega_g = \varepsilon^1 \wedge \dots \wedge \varepsilon^n$ on U.

For existence, we define ω_g locally by declaring $\omega_g := \varepsilon^1 \wedge \dots \wedge \varepsilon^n$ on U. If $(\tilde{\varepsilon}^i)$ a dual coframe to $(\tilde{E}_i)$ oriented, orthonormal frame, on V, can check that on $U \cap V$

$$\tilde{E}_i = A_i^j E_j$$

for some matrix of smooth functions (A_i^j). Since both frames are oriented and orthonormal,

$$(A_i^j(p)) \in SO(n)$$

for all $p \in U \cap V$. In particular, $\det A_i^j(p) = 1$.

Thus, $\omega_g(\tilde{E}_1, \dots, \tilde{E}_n) = \det(A_i^j) = 1 = \tilde{\omega}_g(\tilde{E}_1, \dots, \tilde{E}_n)$ and we get existence of a well-defined form. $\square$

In coordinates:

Prop 15.31

$$\omega_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \dots \wedge dx^n. \quad \square$$

Divergence operator

From now on: instead of writing ω_g for Riemannian volume form, we will write dV_g.

Beware! dV_g is not actually an exact form.!! (Never on a compact manifold)

Several tricks we can play now.

$$\begin{aligned} * : C^\infty(M) &\to \Omega^n(M) \\ f &\mapsto f\, dV_g. \end{aligned}$$

← These are bundle isomorphisms!

$$\begin{aligned} \beta : \mathfrak{X}(M) &\longrightarrow \Omega^{n-1}(M) \\ X &\mapsto X \lrcorner\, dV_g. \end{aligned}$$

Def

$$\begin{tikzcd} \mathfrak{X}(M) \arrow[r, "\mathrm{div}"] \arrow[d, "\beta"'] & C^\infty(M) \arrow[d, "*"] \\ \Omega^{n-1}(M) \arrow[r, "d"'] & \Omega^n(M) \end{tikzcd}$$

The divergence operator of an oriented Riemannian manifold is

$$\begin{aligned} \mathrm{div} : \mathfrak{X}(M) &\to C^\infty(M) \\ X &\mapsto *^{-1}(d(\beta(X))). \end{aligned}$$

Applying Stokes' Theorem, not hard to show

Thm 16.32 (Divergence Theorem) If (M, g) an oriented Riemannian manifold with boundary, then

$$\int_M (\operatorname{div} X)\, dV_g = \int_{\partial M} \langle X, N \rangle_g \, dV_{\tilde{g}}$$

where N is outward pointing unit normal along ∂M and $\tilde{g}$ is the induced Riemannian metric on ∂M. $\square$

Div, grad, curl and all that

Let (M^3, g) be oriented. Can define

$$\begin{aligned} \operatorname{curl} : \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ X &\mapsto \beta^{-1} d(X^\flat). \end{aligned}$$

$$\begin{tikzcd}
C^\infty(M^3) \arrow[r, "\operatorname{grad}"] \arrow[d, equal] & \mathfrak{X}(M) \arrow[r, "\operatorname{curl}"] \arrow[d, "\flat", "\cong"'] & \mathfrak{X}(M) \arrow[r, "\operatorname{div}"] \arrow[d, "\beta", "\cong"'] & C^\infty(M) \arrow[d, "*", "\cong"'] \\
\Omega^0(M^3) \arrow[r, "d"'] & \Omega^1(M) \arrow[r, "d"'] & \Omega^2(M) \arrow[r, "d"'] & \Omega^3(M)
\end{tikzcd}$$

Commuting diagram! Thus, $\operatorname{div} \circ \operatorname{curl} = 0$, $\operatorname{curl} \circ \operatorname{grad} = 0$.

~~Hodg~~ Hodge Star (M, g) oriented.

$$\begin{tikzcd}
\Omega^0(M^n) \arrow[r, "d"] \arrow[d, equal] & \Omega^1(M) \arrow[r] \arrow[rr, bend right, "\vdots\ *"'] & \cdots \arrow[r] & \Omega^{n-1}(M) \arrow[r] & \Omega^n(M) \\
C^\infty(M^n) \arrow[urrrr, bend right, "*"'] & & & &
\end{tikzcd}$$