Lecture 16.1

Reading: first part of Milnor's *Morse Theory*.

Announcements:

- please do course evaluations!

Practice: 13-24

This week: the many guises of the Euler characteristic: topological, analytic and geometric!

Def For a smooth (compact) manifold M of dimension n, the Euler characteristic is

$$\chi_{dR}(M) := \sum_{i=0} (-1)^i \dim H^i_{dR}(M) \in \mathbb{Z}.$$

Def For ~~a simplicial complex~~ any topological space X, its Euler characteristic is

$$\chi_{sing}(X) := \sum_{i=0}^{\infty} (-1)^i \operatorname{rk}_{\mathbb{Z}} H_i^{sing}(X;\mathbb{Z}) \in \mathbb{Z}$$

De Rham's theorem (+ Universal coefficient theorem): $H^i_{dR}(M) \cong \operatorname{Hom}_{\mathbb{Z}}(H_i^{sing}(X;\mathbb{Z}), \mathbb{R})$

In particular: $\dim H^i_{dR}(M) = \operatorname{rk}_{\mathbb{Z}} H_i^{sing}(M;\mathbb{Z})$.

$$\implies \chi_{dR}(M) = \chi_{sing}(M).$$

We will then just write $\chi(M)$.

If X is homeomorphic ~~(in fact, homotopy equivalent suffices)~~ to a simplicial complex Δ, then X is called <u>triangulable</u>. For triangulable spaces, we can use Δ to compute $\chi(X)$ very easily:

$$\chi(X) = \chi(\Delta) = \sum_{i=0}^{\infty} (-1)^i \#\Delta_i,$$ where Δ_i is the set of i-dimensional simplices.

follows by "simplicial approximation" i.e. $H_i^{sing}(\Delta) \cong H_i^{simp}(\Delta)$

Whitehead (1940): every smooth manifold ~~admits~~ is triangulable.

by end of class Wed, we will have a new proof of Hairy Ball Theorem from this.

$\underline{\text{Ex}}$ $\chi(\cdot) = 1$, $\chi(S^1) = 0$, $\chi(S^2) = 2$

$\chi($  $) = 0$, $\chi(\text{orientable closed 3-manifold}) = 0$

Poincaré duality.

$$\chi(S^n) = \begin{cases} 0 & \text{if } n \text{ odd} \\ 2 & \text{if } n \text{ even} \end{cases}$$

Today: Sketch yet another approach to $\chi(M)$ using Morse theory.

Morse functions

Def A critical point p of $F: M \to \mathbb{R}$ is called non-degenerate if in any (hence, all) coordinate charts around p, the Hessian

$$\left(\frac{\partial^2 F}{\partial x^i \partial x^j}(p) \right)$$

is ~~...~~ invertible.

More intrinsically:

Def The Hessian of a critical point p of $F: M \to \mathbb{R}$ is the symmetric bilinear form

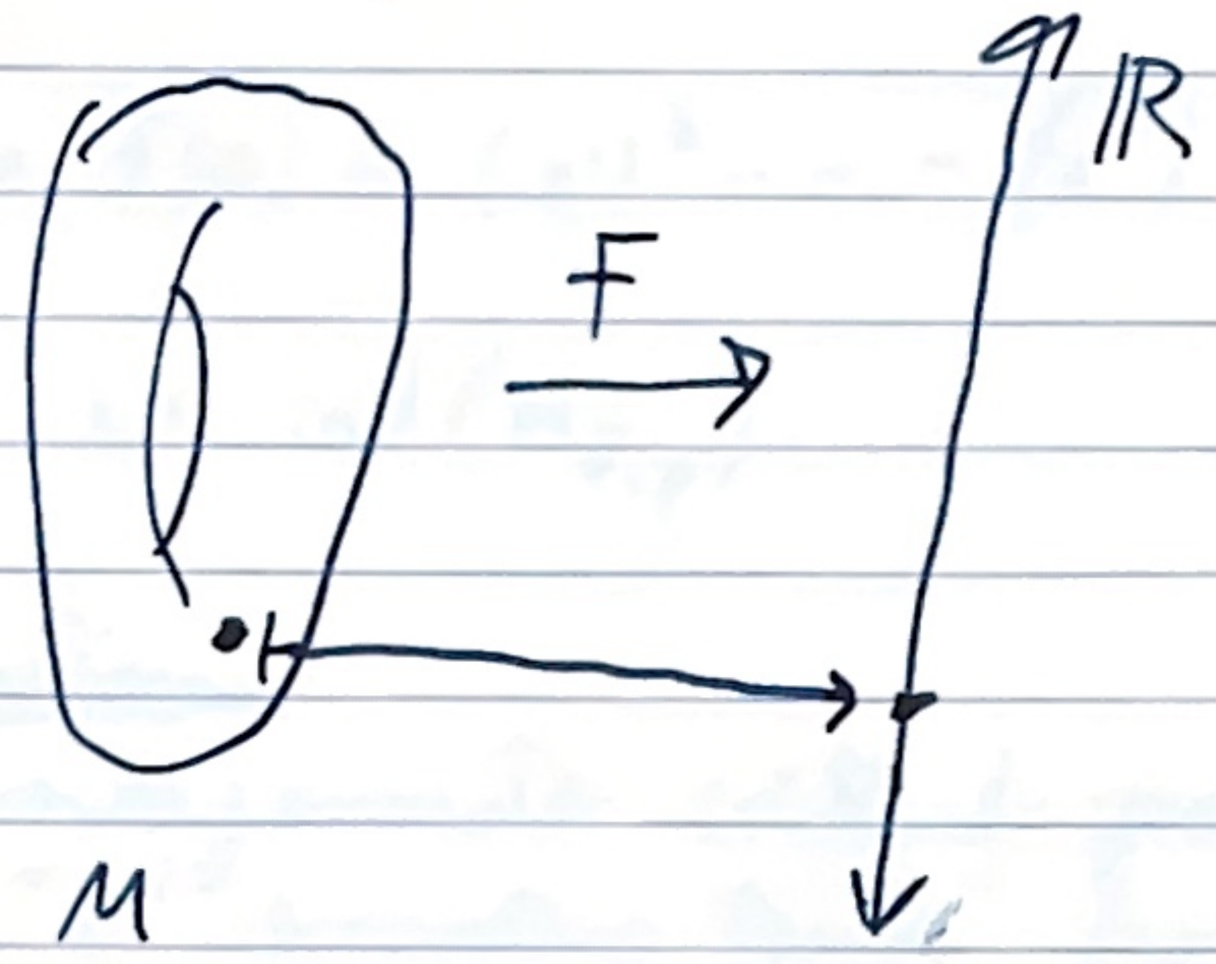
$$H_{F,p} : TM_p \times TM_p \to \mathbb{R}$$
$$(v, w) \mapsto v(\tilde{w}(F))$$

where $\tilde{w}$ is any smooth vector field extending w to a neighborhood of p.

Ex: This is well-defined independant of $\tilde{w}$
(Hint: show it is symmetric first!)

Def a ^smooth^ function $F: M \to \mathbb{R}$ is called **Morse** if it has no degenerate critical points.

Ex Height function on torus:



Non-examples: the critical points of following functions are all degenerate:

(a) $F(x) = x^3$
(b) $F(x) = e^{-1/x^2} \sin^2(1/x)$
(c) $F(x,y) = x^3 - 3xy^2$ "monkey saddle"
(d) $F(x,y) = x^2$
(e) $F(x,y) = x^2 y^2$

Def Let $B: V \times V \to \mathbb{R}$ be a symmetric bilinear form. Then the index of B is

$$\operatorname{ind}(B) := \max\{\dim W \mid W \leq V \text{ s.t. } B|_{W \times W} \text{ negative definite}\}$$

$= \#\{\text{negative eigenvalues of } B\}$

The nullity of B is

$$\operatorname{null}(B) := \dim\{v \in V \mid B(v,w) = 0 \ \forall w \in W\}$$
$$= \dim \operatorname{Ker}(v \mapsto B(v,-)).$$

Note: a ~~nondegenerate~~ critical point p of $F: M \to \mathbb{R}$ is nondegenerate iff $\operatorname{null}(H_{F,p}) = 0$, and $0 \leq \operatorname{ind}(H_{F,p}) \leq n = \dim M$.

Morse Lemma Let p be a non-degenerate critical point of $f: M \to \mathbb{R}$. Then there exists a regular coordinate neighborhood centered at p w/ coordinates $x^1, \dots, x^n$ such that

$$f = f(p) - (x^1)^2 - \dots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \dots (x^n)^2$$

where $\lambda = \operatorname{ind}(H_{f,p})$. $\square$

Observations:
- nondegenerate critical points are isolated.
- "2^{nd} derivative test" holds for non-degenerate critical points:

  p is a local min iff $\operatorname{ind}(H_{f,p}) = 0$

  p is a local max iff $\operatorname{ind}(H_{f,p}) = n$

Sard's theorem can be used to show:

Theorem Let $M \in \mathbb{R}^k$ be embedded. Then for almost all $p \in \mathbb{R}^k$, the function

$$L_p : M \to \mathbb{R}$$
$$q \mapsto \|q - p\|^2$$

is Morse. $\square$

Applying Whitney embedding:

Corollary Every smooth manifold admits a Morse function. $\square$

Handle decompositions from Morse functions

A Morse function on a manifold can be used to decompose the manifold into some simple pieces (called "handles").

Fix a Morse function $f: M \to \mathbb{R}$. Let

$$M_a := f^{-1}(-\infty, a] = \{p \in M \mid f(p) \le a\}$$

"sub-level set."

<u>Thm</u> Suppose f has no critical value in $[a,b]$ and M is compact. Then $M_a \cong M_b$, and M_b deformation retracts onto M_a.

<u>Proof</u>: Pick a Riemannian metric g on M. Let X be a vector field supported on a compact neighborhood of $f^{-1}[a,b]$ for which

$$X_q = \frac{\operatorname{grad} f}{g(\operatorname{grad} f, \operatorname{grad} f)}$$

for all $q \in f^{-1}[a,b]$. (Use bump function.) Let $\varphi_t: M \to M$ be the global flow generated by X. Then $\varphi_{b-a}|M_a$ is a diffeomorphism onto M_b. Use the flow to build the deformation retraction --- $\square$

<u>Thm</u> Let $c = f(p)$ and pick $\varepsilon > 0$ so that p is only critical ~~value~~ point in $f^{-1}[c-\varepsilon, c+\varepsilon]$. If ε is small enough, then

$$M^{c+\varepsilon} \simeq M^{c-\varepsilon} \coprod_{\partial \mathbb{B}^\lambda \to M^{c-\varepsilon}} \mathbb{B}^\lambda$$

where $\lambda = \operatorname{ind}(H_{f,p})$. $\square$ (Draw picture for torus.)

<u>Thm</u> If M compact, then it has the homotopy type of a CW-complex w/ one cell of dimension λ for each critical point of index λ. $\square$ (See Milnor)

In particular, using $H^*_{CW} \cong H^*_{Sing}$, get

$$\chi(M) = \sum_{\lambda=0}^{n} (-1)^\lambda C_\lambda \quad (*).$$

"Morse homology"

where $C_\lambda := \#\{\text{critical points of } f \text{ w/ } \operatorname{ind}(H_{f,p}) = \lambda\}$