Lecture 16.2

Reading: §6 of Milnor's "Topology from the Differential Viewpoint"

Today: Computing $\chi(M)$ "analytically" using any vector field on M with isolated zeros.

Let X be a smooth vector field on M^n, with $p \in M^n$ an isolated 0 of X, (meaning $\exists$ open neighborhood of p where $X_q = 0 \Rightarrow q = p$). Fix a regular coordinate ball (U, φ) around p, with $\varphi : U \xrightarrow{\cong} B^n$ extending to a smooth diffeo $\bar{\varphi} : \bar{U} \to \bar{B}^n$. ~~Then the index of X at p is the degree of the map~~ ~~use $\bar{\varphi}$ to pullback~~

$$\cancel{S^{n-1} = \partial \bar{B}^n \xrightarrow{\bar{\varphi}^{-1}} \bar{U}}$$

In these coordinates, X yields a nonvanishing vector field $X|_{S^{n-1}} : S^{n-1} \to TS^{n-1} \subseteq S^{n-1} \times \mathbb{R}^n$.

$$p \mapsto (p, d\bar{\varphi}_{\bar{\varphi}^{-1}}(X_{\bar{\varphi}^{-1}(p)})).$$

Define $Y : S^{n-1} \to S^{n-1} \subseteq \mathbb{R}^n$

$$p \mapsto \frac{X_p}{\|X_p\|}.$$

Def The index of X at p is the degree of Y_1 denoted $\text{ind}_X(p)$

Ex This definition is well-defined independent of the regular coordinates (U, φ) we chose around p.

Poincaré-Hopf Index Theorem

If M is a compact manifold and X is a vector field with isolated zeroes, then

$$\sum_{\substack{p \in M \\ X_p = 0}} \text{ind}_X(p) = \chi(M).$$

Corollary Even dimensional coconuts can't be combed.

[Holds if $\partial M \neq \phi$, but need X to point outward from ∂M.]

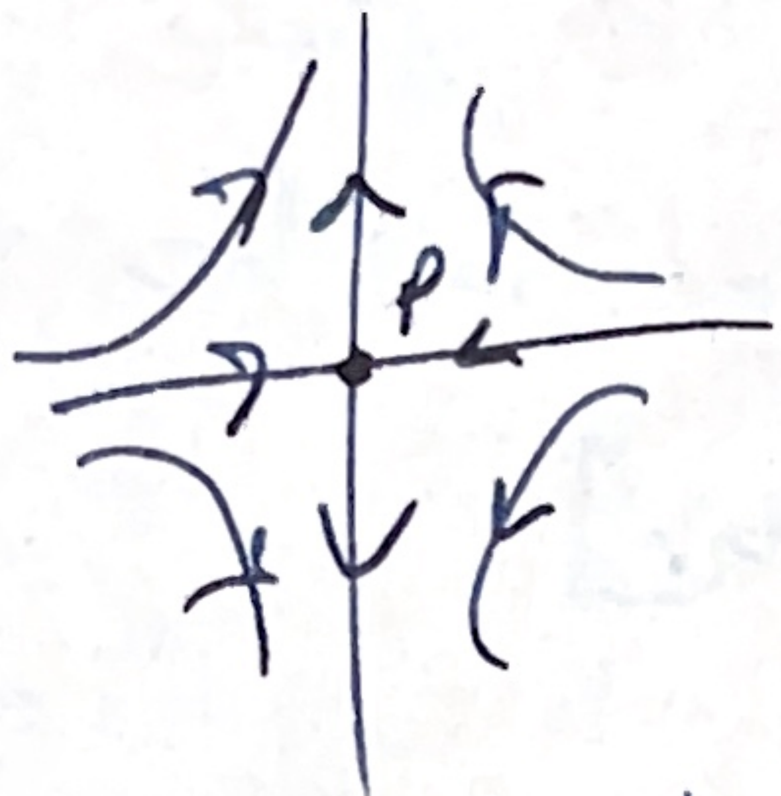
Proof has two parts:

I. Show that for at least one X, it's true.

II. Show that $\sum \text{ind}_X(p)$ does not depend on X.

I'll give two proofs of I. To prove II, we will use an ~~strengthening~~ improvement of degree theory that allows for disconnected ~~regions~~ domains.

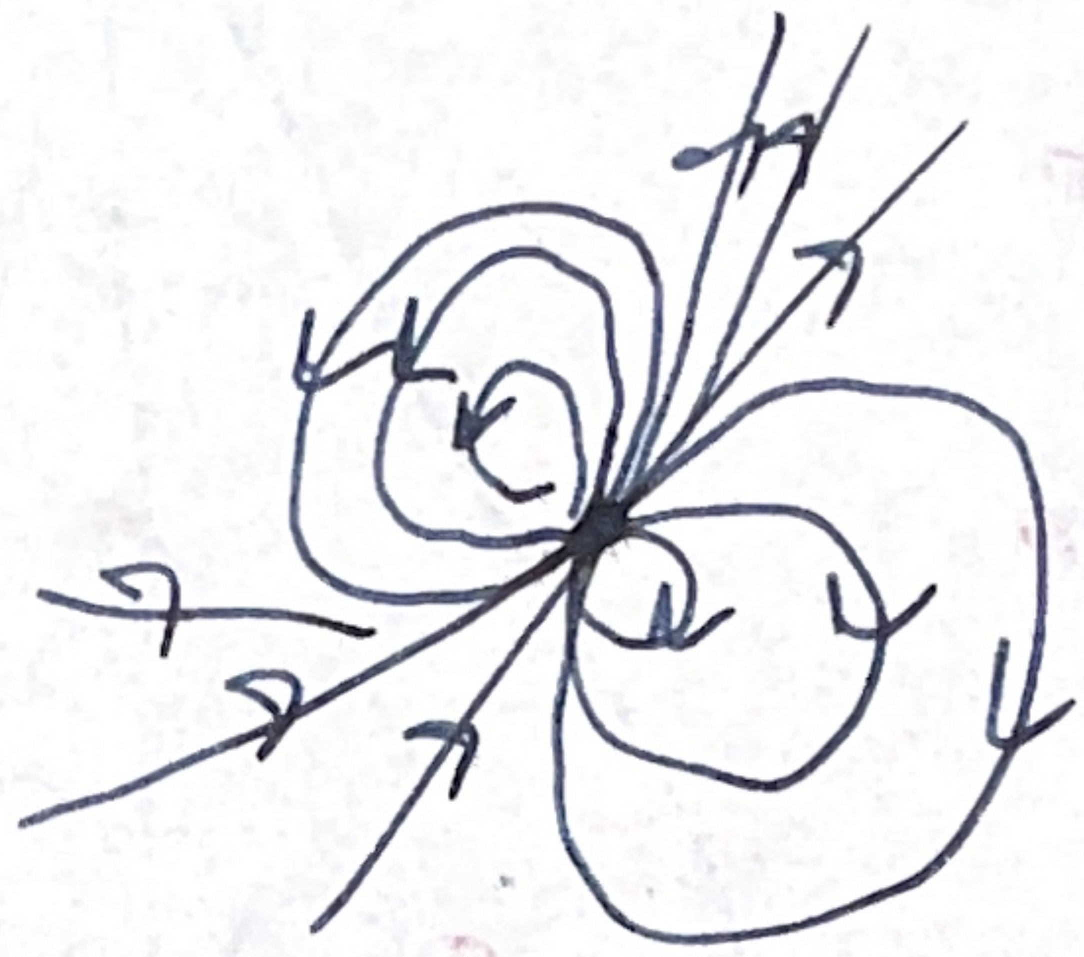
Examples $n=2$: "winding number"



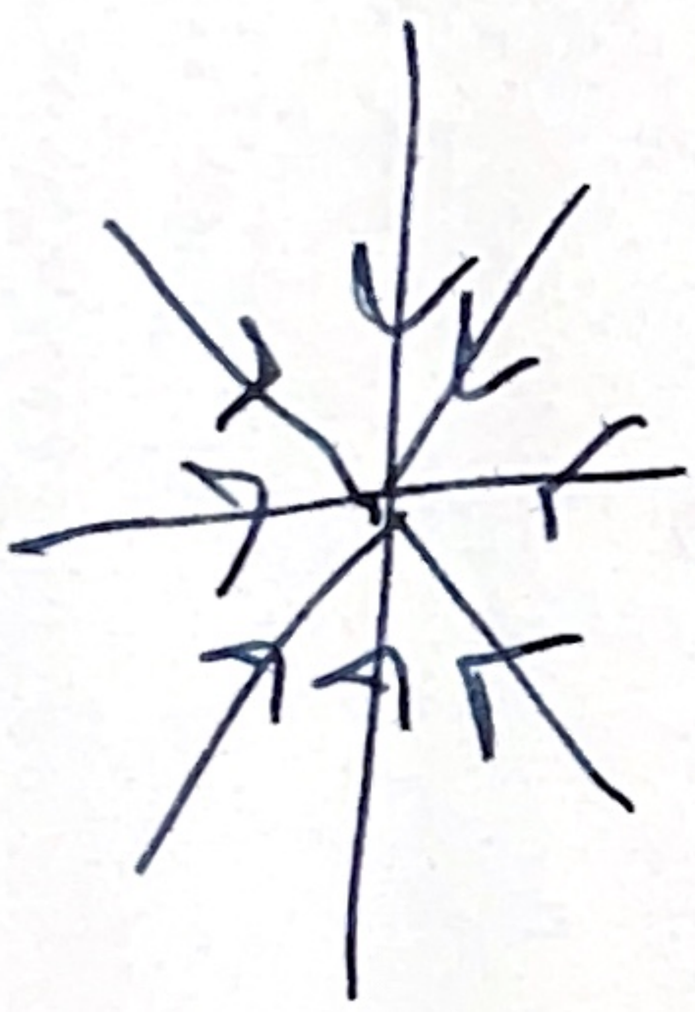
$\mathrm{ind}_X(p) = -1$

$\mathrm{ind}_X(p) = 0$

$\mathrm{ind}_X(p) = -1$



$\mathrm{ind}_X(p) = +2.$



$\mathrm{ind}_X(p) = +1$

$\mathrm{ind}_X(p) = +1$

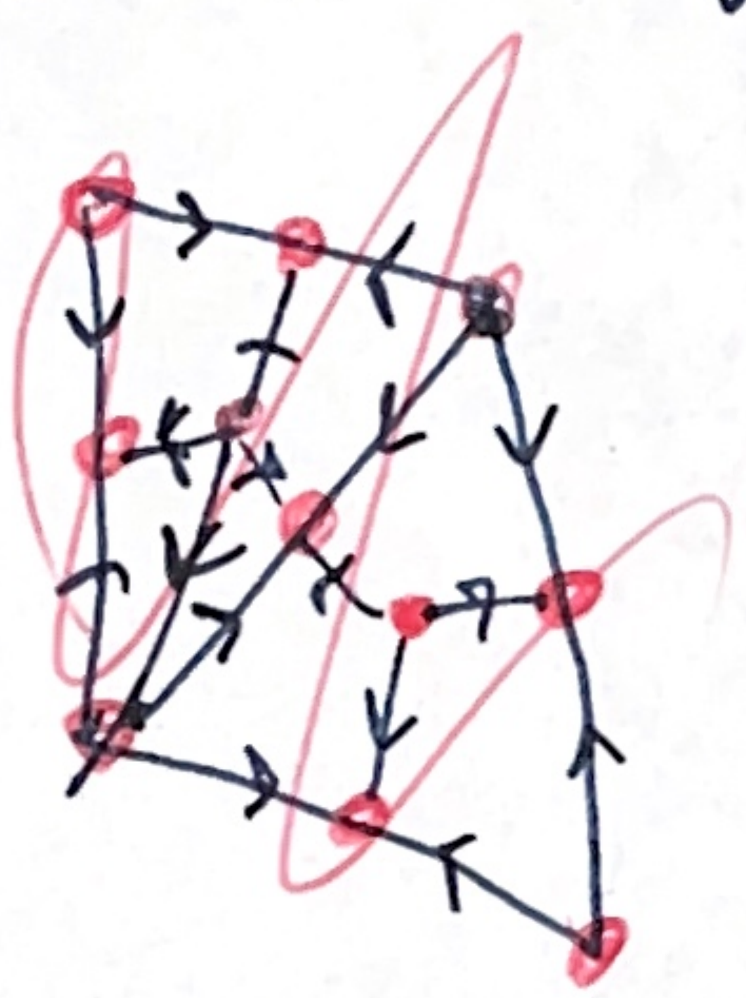
$n=1$:

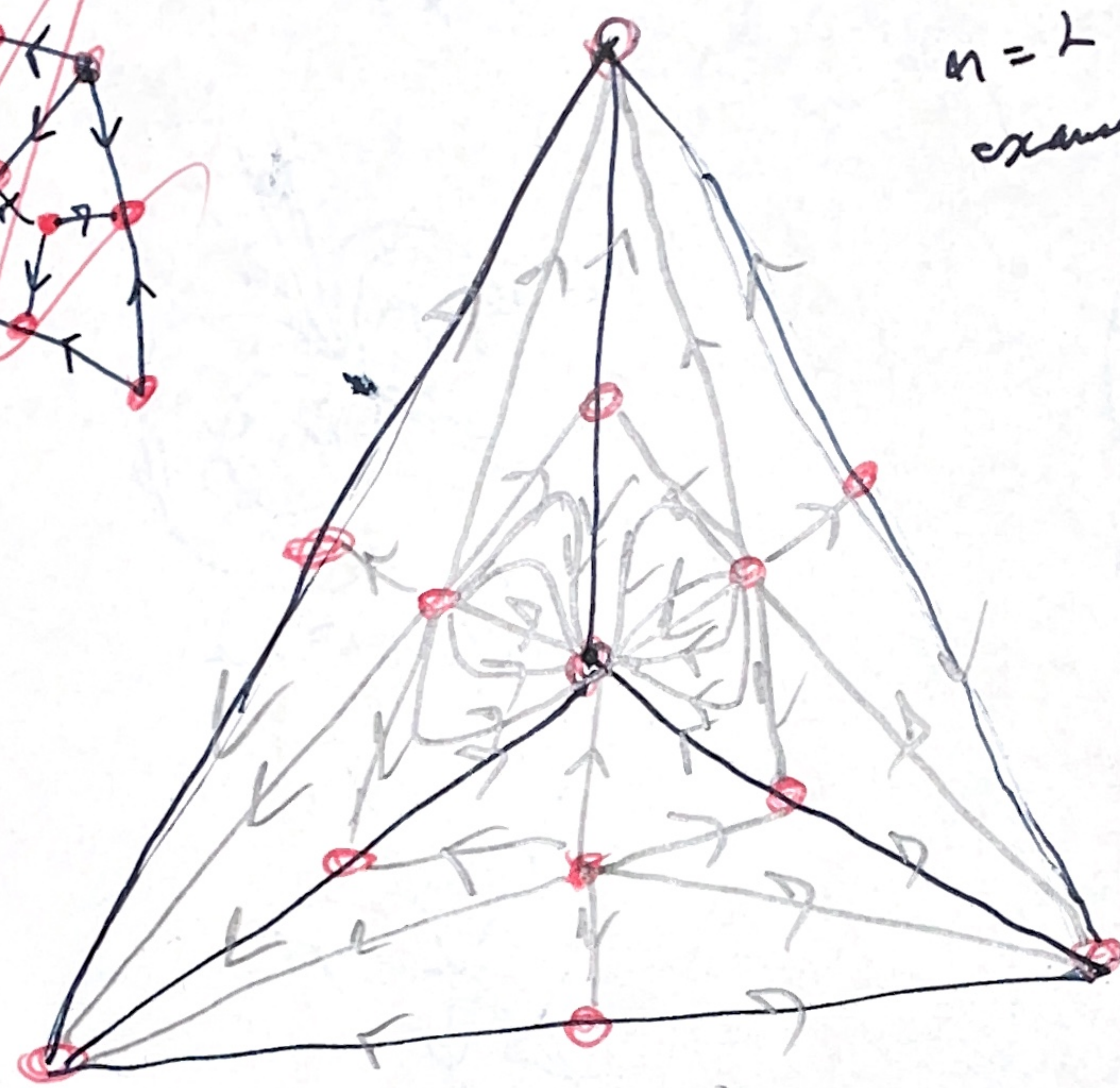
$\mathrm{ind}_X(p) = -1$

$\mathrm{ind}_X(p) = +1$

Note: as an oriented manifold, standard is that $S^0 = \overset{+}{\bullet}\ \overset{-}{\bullet}$

I A. Recall Whitehead's theorem, which says every smooth manifold is triangulable. Build a vector field X on M that has a zero of index $(-1)^k$ in the middle of each k-simplex of a triangulation Δ





n = 2 example

etc...

Thus, $$\sum_p \mathrm{ind}_X(p) = \sum_{k=0}^{n} (-1)^k \, \#\{k\text{-simplices of } \Delta\}$$

✓

IB. Let $F: M \to \mathbb{R}$ be a Morse function, and pick a Riemannian metric on M. Then grad F is a vector field w/ isolated zeroes. As mentioned quickly at end of class Monday, can use F to identify M (up to homotopy equivalence) as a bunch of disks of different dimension pasted together (yielding a "CW-complex"):

$$M \simeq \left(\bigsqcup_{p:\text{ critical point of } F} \overline{B}^{\,\mathrm{ind}(H_{F,p})} \right) \Big/ \sim$$

$$\Rightarrow \chi(M) = \sum_{p:\text{ critical point of } F} (-1)^{\mathrm{ind}(H_{F,p})}$$

$$= \sum_{p:\text{ zero of grad } F} (-1)^{\mathrm{ind}(H_{F,p})}$$

Check!

$$= \sum_{p:\text{ zero of grad } F} \mathrm{ind}_{\mathrm{grad}\, F}(p)$$

✓

II.

Def Let $F: M \to N$ be a ~~smooth~~ continuous map between oriented compact manifolds where N is connected, and $M = \bigsqcup_{i=1}^{l} M_i$ has l connected components. Then

$$\deg F := \sum_{i=1}^{l} \deg F|_{M_i}.$$

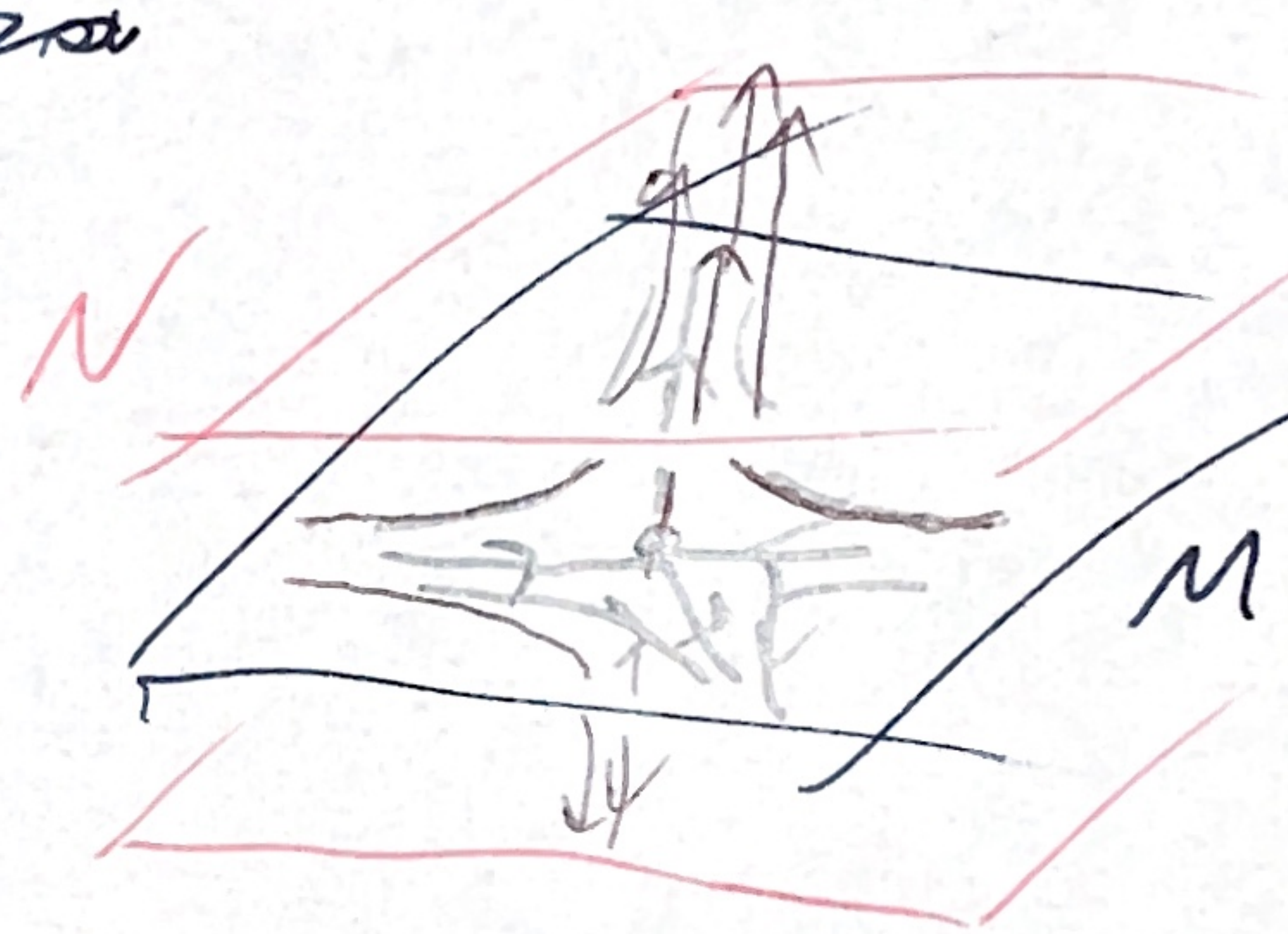
Crucially, Theorem 17.38 works in this more general setting (anyone remember me mumbling something about this?)

Thm 17.38 Suppose N connected, compact, oriented, smooth n-manifold and X is a compact, oriented, smooth $(n+1)$-manifold ~~w/ connected boundary~~ not needed! . If $F: \partial X \to N$ is continuous and has an extension to X, then $\deg F = 0$.

careful w/ NOTATION!

We're going to show that for any vector field X w/ isolated 0's on M, the index sum equals the degree of a map that does not depend on X.

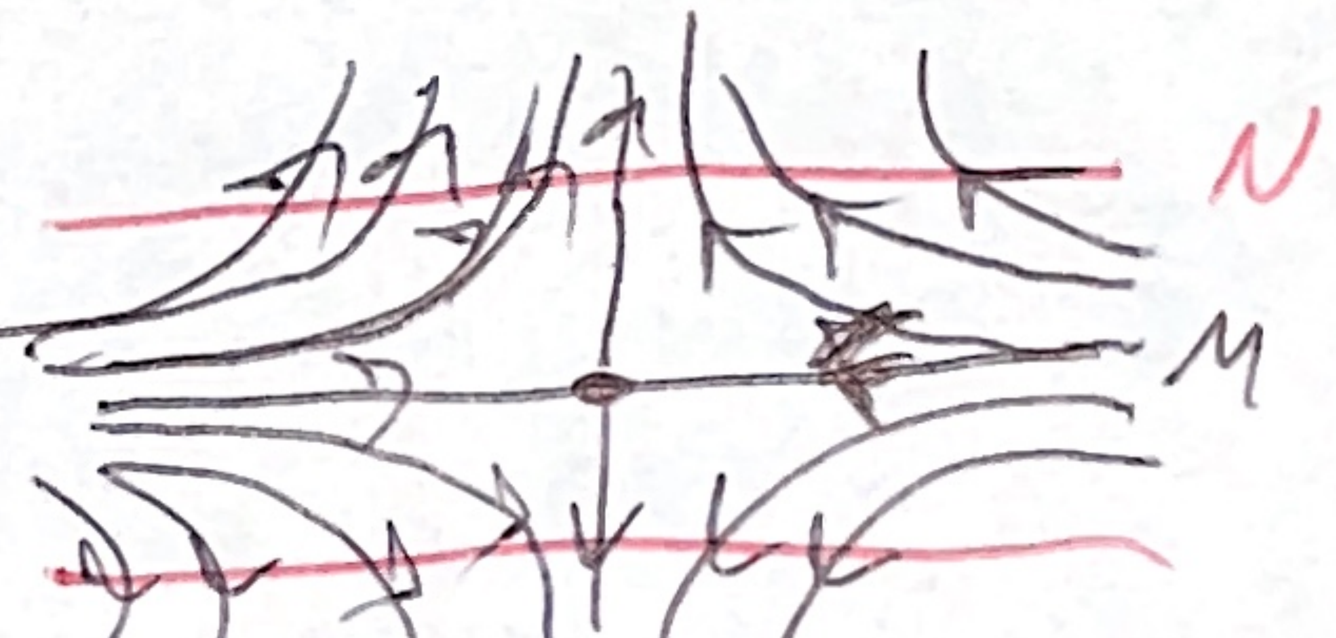
Using Whitney, embed M into $\mathbb{R}^d$ for some d large enough. Let N be a closed tubular neighborhood of M. Notice that N is compact, ~~and~~ orientable, with boundary. (Why?) We can extend X to a vector field $\tilde{X}$ on all of N ~~in a manner~~ ~~way that~~ so that:



(i) zeros of $\tilde{X}$ are same as zeros of X

(ii) $\mathrm{Ind}_X(p) = \mathrm{Ind}_{\tilde{X}}(p)$

(iii) $\tilde{X}$ is outward pointing.



~~In particular.~~

Let P be N w/a ~~small~~ regular ball removed ~~from~~ around each zero of $\tilde{X}$. Define

$$\tilde{Y}: P \to S^{d-1}, \quad p \mapsto \frac{\tilde{X}_p}{\|\tilde{X}_p\|}.$$

~~Note~~ Thus 17.35 implies

$\deg \tilde{Y}|_{\partial P} = 0$.

However, $\partial P = \partial N \sqcup \bigsqcup_{p} \partial(\text{regular coordinate ball at } p)$

$$\Rightarrow \sum \mathrm{ind}_X(p) = \sum \mathrm{ind}_{\tilde{X}}(p) = \deg\left(\begin{array}{l} \partial N \to S^{d-1} \\ q \mapsto \tilde{X}_q / \|\tilde{X}_q\| \end{array}\right)$$

Does NOT depend on X! $\square$