Lecture 16.3

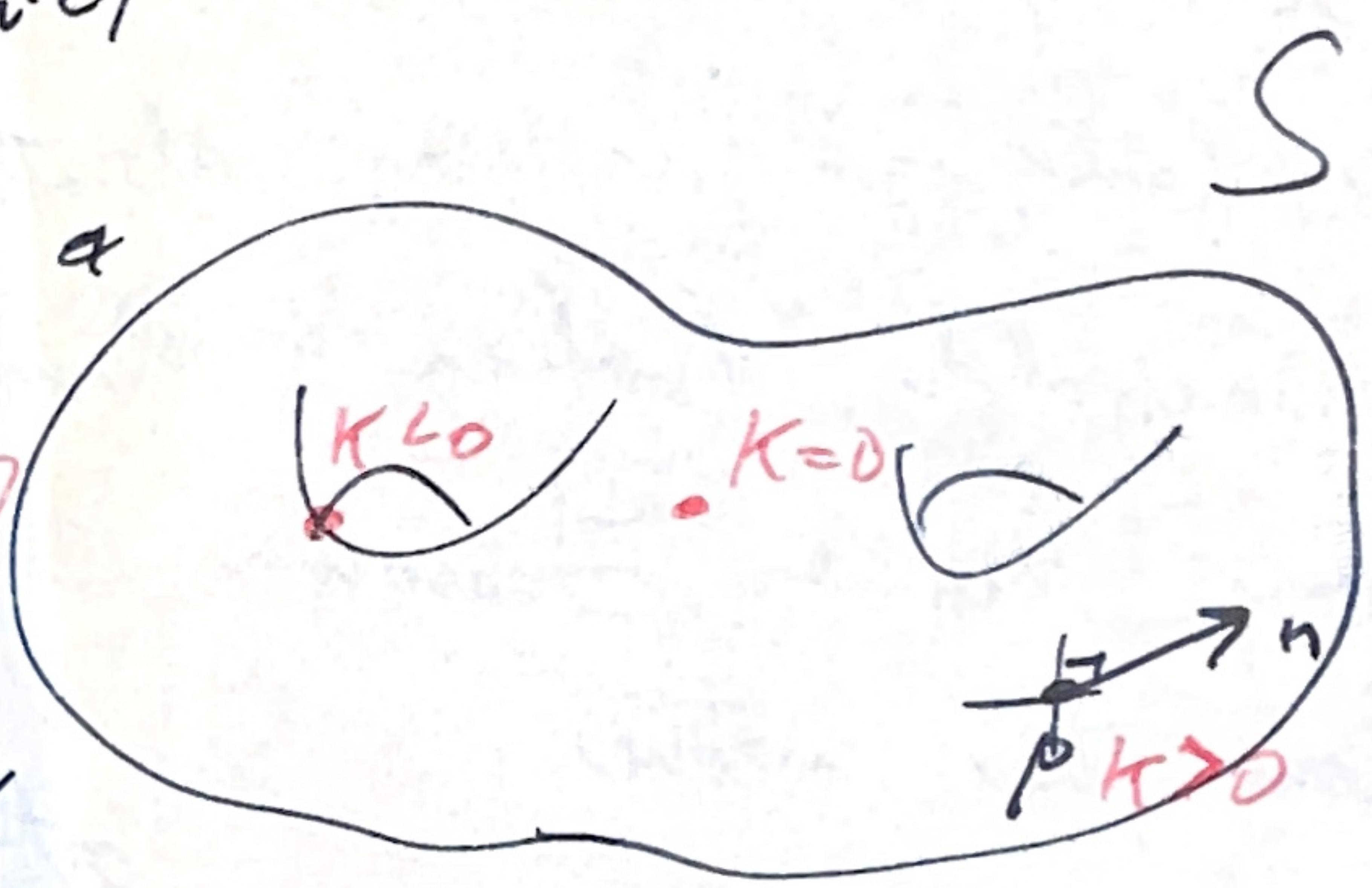
Today: use Poincaré-Hopf to prove Gauss-Bonnet.

Please do course evaluations!

Gaussian curvature: extrinsic definition

Let S be a properly embedded 2-manifold ~~surface~~ in $\mathbb{R}^3$ (i.e., an embedded surface).

Let $p \in S$ and pick a normal vector n to S at p. A plane $V \subseteq \mathbb{R}^3$ is called normal if $n \in V$.



Orient S by picking a nowhere vanishing normal vector field, which we might as well assume is unit length. Such a vector field is called a Gauss map for the embedded surface:

(note: S has exactly two Gauss maps if it has any.)

$$n : S \to S^2.$$

Def The Gaussian curvature of S at p is $\det dn_p =: K(p)$.

Gauss's Theorem Egregrum

If we equip S as above with the Riemannian metric it inherits from $\mathbb{R}^3$, then K is a local isometry invariant. □

Explain why we fold pizza.

~~Gaussian curvature: intrinsic definition~~

~~**Def** If~~

In fact, given any abstract 2-dim'l Riemannian manifold ~~(called a Riemann surface)~~, one can define Gaussian curvature intrinsically from the metric.

Let e_1, e_2 be a local orthonormal frame for (S, g) and ω^1, ω^2 the dual frame.

Claim $\exists!\ \omega^{12}$ such that

$$d\omega^1 = \omega^{12} \wedge \omega^2 \quad \text{and} \quad d\omega^2 = -\omega^{12} \wedge \omega^1.$$

Proof Take

$$\omega^{12} = d\omega^1(e_1, e_2)\,\omega^1 - d\omega^2(e_1, e_2)\,\omega^2. \quad \square$$

ω^{12} called connection 1-form in the frame $\{e_1, e_2\}$. (closely related to the "Levi-Civita connection" of g)

Claim $d\omega^{12} = -K\,\omega^1 \wedge \omega^2 = -K\,dV_g$.

We will not prove this. (Take it as a definition)

We are now ready to prove:

<u>Gauss-Bonnet Theorem</u>

Let (S, g) be a 2-dim'l Riemannian manifold that is oriented, compact, without boundary. Then

$$\int_S K\,dV_g = 2\pi\,\chi(S).$$

<u>Proof</u> We will use Poincaré-Hopf. Let X be a vector field on S w/ isolated zeroes, $x_1, \dots, x_k \in S$. Use X and g to define the unit vector field e_1 on $S - \{x_1, \dots, x_k\}$ by $e_1(p) := X_p / \|X_p\|$. Use the orientation on S to define $e_2(p)$ to be the oriented orthonormal to $e_1(p)$. Then e_1, e_2 is an oriented orthonormal frame on $S - \{x_1, \dots, x_k\}$.

Let $U_1, \dots, U_k$ be disjoint regular coordinate balls centered at $x_1, \dots, x_k$.

Each $U_i \cong B^2$ and thus has an oriented orthonormal ~~so that $e_1^{(i)}$ is outward normal.~~ frame $e_1^{(i)}, e_2^{(i)}$.

$$\int_S K\,dV_g = \int_{S-U_i} K\,dV_g + \sum_{i=1}^{k} \int_{\bar{U}_i} K\,dV_g$$

$$= \int_{S-U_i} K\,\omega^1 \wedge \omega^2 + \sum_{i=1}^{k} \int_{\bar{U}_i} K\,\omega^1_{(i)} \wedge \omega^2_{(i)}$$

$$= -\int_{S-U_i} d\omega^{12} - \sum_{i=1}^{k} \int_{\bar{U}_i} d\omega^{12}_{(i)}$$

Stokes

$$= -\int_{-\bigcup \partial \bar{U}_i} \omega^{12} - \sum_{i=1}^{k} \int_{\partial \bar{U}_i} \omega^{12}_{(i)}$$

$\partial(S-U_i) = -\bigcup \partial \bar{U}_i$

$$\boxed{\cancel{\partial S = -\bigcup_{i=1}^{k} \partial \bar{U}_i}}$$

~~w/out this~~

$$= \sum_{i=1}^{k} \int_{\partial \bar{U}_i} \omega^{12} - \omega^{12}_{(i)}$$

For each i, let $\tau_i : \partial \overline{U}_i \to SO(2) = S^1 = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| a^2+b^2=1 \right\}$

$$p \mapsto \tau_i(p).$$

where $\tau_i(p) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ defined by

$$e_1^{(i)} = a e_1 + b e_2, \quad e_2^{(i)} = -b e_1 + a e_2$$

$$\Longrightarrow w^1_{(i)} = a w^1 + b w^2, \quad w^2_{(i)} = -b w^1 + a w^2$$

$$\vdots$$

$$\underset{\text{compute}}{\Longrightarrow} \quad w^{12} - w^{12}_{(i)} = \tau_i^* d\theta$$

$$\Rightarrow \int_{\partial \overline{U}_i} w^{12} - w^{12}_{(i)} = \int_{\partial \overline{U}_i} \tau_i^* d\theta = \deg \tau_i \int_{S^1} d\theta = 2\pi \deg \tau_i$$

$$\Longrightarrow \int_S K \, dV_g = 2\pi \sum_{i=1}^{k} \deg . \tau_i .$$

~~Note that $\deg \tau_i$ is exactly counting how many times e_1 winds around~~

<u>Check</u>: $\deg \tau_i = \mathrm{ind}_{e_1}(x_i) - \mathrm{ind}_{e_1^i}(x_i) = \mathrm{ind}_{e_1}(x_i).$

$$= \mathrm{ind}_X(x_i)$$

0 b/c e_1^i extends to a non-vanishing vector field on $\overline{U}_i$

$$\Rightarrow \int_S K \, dV_g = 2\pi \sum_{i=1}^{k} \mathrm{ind}_{e_1}(x_i) \underset{\text{P-H}}{=} 2\pi \chi(M). \qquad \square$$