CS 593/MA 592 - Intro to Quantum Computing
Spring 2024
Tuesday, April 23 - Lecture 12.1

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1 Classical and Quantum Error Correction

Suppose we want to send a single bit, but our connection channel is noisy. See binary symmetric channel in Fig. 1. Can I reliably send you my message?

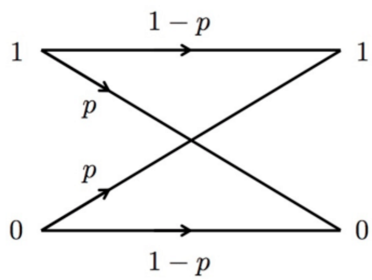

Figure 1: Binary symmetric channel.

Our classical code to do so is called the repetition code. We will do the repetition code on 3 ”physical” bits.

$$\{000, 111\} \subseteq \{000, 001, ..., 111\} \tag{1}$$

Encoding is as following:

$$0 \mapsto 000 \tag{2}$$
$$1 \mapsto 111 \tag{3}$$

Decoding is as following:

$$\{000, 001, 010, 100\} \mapsto 0 \tag{4}$$
$$\{011, 101, 110, 111\} \mapsto 1 \tag{5}$$

Prob (2 or more bits in my encoded message are flipped)

$$= 3p^2(1-p) + p^3 = p^2(3-2p) \tag{6}$$

As long as $p < 1/2$, this decoder is more likely to get the correct message than the wrong one.

More-or-less all error corrections -both quantum and classical- works like the repetition code, i.e by storing info in a redundant fashion.

Both classical error correction and quantum error correction run into ”real world” feasibility questions. How do we determine the correct model? Quantumly, several new issues arise that are not present classically.

- No cloning.
- There are continuously many possible quantum errors.

- Measurement causes state collapse.

Fortunately, all of these can be dealt with in theory.
My 2 cents: Discretizing is the key issue.
There is a fourth issue, which we will ignore for now: - Decoding (and encoding) requires measurement, which is a new source of error.
There is essentially the fault-tolerance problem. For now, the types of measurements we will need for decoding will be assumed "not that crazy" and error free.

1.1 Bit flip code:

Naive generalization of classical repetition code. Not a very good QEC.
It will protect against quantum bit flip errors.
One kind of error is "X error", which happens as follow:

$$|0\rangle \mapsto |1\rangle \tag{7}$$
$$|1\rangle \mapsto |0\rangle \tag{8}$$

We will do a specific example on 3 "physical" qubits.

Code: $C = span[|000\rangle, |111\rangle] \subseteq (\mathbb{C})^{\otimes 3}$. $dimsC = 2$. So, we might say that C encodes 1 logical qubit.

Encoding: $|\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle$ does not make sense because of no cloning. It needs a map $\mathbb{C}^2 \mapsto C \leq (\mathbb{C}^2)^{\otimes 3}$.

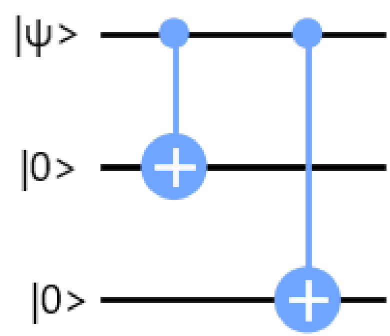

Figure 2: Quantum circuit for encoding.

For quantum circuit see Fig. 2. If $|\psi\rangle = a|0\rangle + b|0\rangle$, then this circuit outputs $a|000\rangle + b|111\rangle$. We aim to show that there exists a way to detect and correct single X errors on the physical.
Warning: We cannot simply read off all the qubits, since that will destroy most of the states $|\psi\rangle \in C$ we hope to protect.
E.g. If $|\psi\rangle = |000\rangle + |111\rangle = |0_L\rangle + |1_L\rangle$ and we measure all of qubits in computational basis will have collapses. Instead, we will perform measurements in pairs called "syndrome measurements" (or stabilizer checks) on pairs of qubits.
Consider the following hermitian operators:

$$P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$$

$$P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$$
$$P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$$

Each of the projective measurements associated to those have two outcomes; 0 and 1.

Suppose $|\psi\rangle \in (\mathbb{C}^2)^{\otimes 3}$ is either in C or looks like

$$X_1|\psi'\rangle = (X \otimes Id \otimes Id)|\psi'\rangle$$
$$X_2|\psi'\rangle = (Id \otimes X \otimes Id)|\psi'\rangle$$
$$X_3|\psi'\rangle = (Id \otimes Id \otimes X)|\psi'\rangle$$

for some $|\psi'\rangle \in C$.

When we perform measurement P_i on such a $|\psi\rangle$, we will make the following interpretation.

Outcome: $0 \mapsto$ no X_i error.

Outcome: $1 \mapsto X_i$ error.

For a $|\psi\rangle$ as above, the process of performing these three measurements is called ”error diagnosis” or ”syndrome extraction”.

We will perform ”recovery” step based on the error syndrome that will allow us to correct a single X error.

Note that for the kind of $|\psi\rangle$ we are interested in the P_i measurements that does not collapse $|\psi\rangle$.

E.g. Performing P_1 on $a|100\rangle + b|011\rangle$; (eigenvalue 1) returns $a|100\rangle + b|011\rangle$.

For recovery stage; For each P_i that reports outcome 1, we will simply apply X_i.

Bad news: The bit flip code cannot detect any single Z error.

Given

$$|\psi\rangle = a|000\rangle + b|111\rangle \in C$$
$$Z_2|\psi\rangle = a|000\rangle - b|111\rangle \in C$$

We can NEVER detect an error that takes code space to itself. We will perform error syndrome that correct a single X error.

1.2 Phase flip code on 3 physical qubits

Intuition: Rather than the repetition in computation basis use $|+\rangle$, $|-\rangle$ basis.

Code:

$$C = span\{|+++\rangle, |---\rangle\} \leq (C^2)^{\otimes 3}$$

Encodes: For quantum circuit see Fig. 3.

Like for bit flip code, this can detect and correct any single qubit Z error, but NO X errors.

1.3 Shor q-qubit code

Concatenetion of 3-qubit phase flip code with 3-qubit bit flip code:

Code:

$$C = span\{|O_L\rangle, |1_L\rangle\} \leq (\mathbb{C}^2)^{\otimes q}$$

Encode: For quantum circuit see Fig. 4.

$$|0_L\rangle = \frac{1}{2\sqrt{2}}[(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)]$$

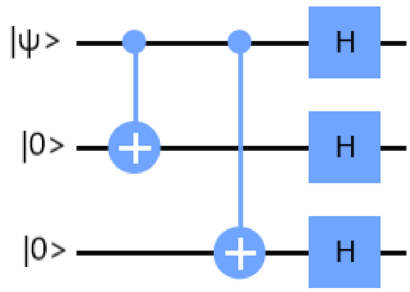

Figure 3: Quantum circuit for encoding.

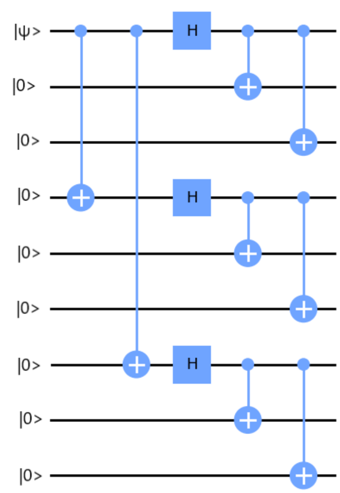

Figure 4: Quantum circuit for encoding.

$$|1_L\rangle = \frac{1}{2\sqrt{2}}[(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)]$$

Claim: Shor's q-qubit code can detect and correct any single X or Z error on any of the q physical. In fact it can correct any single qubit error.