CS 593/MA 592 - Intro to Quantum Computing
Spring 2024
Thursday, April 4 - Lecture 12.2

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Reading: Section 2.4 of Nielsen and Chuang.

Agenda:

1. Informal definition to quantum operation (quantum channel)
2. Briefly motivating why we introduce the mixed states to study the quantum error correction
3. Definitions to ensemble, mixed state and density operator
4. Behavior of projective measurement on mixed states.

1 Motivation

In the following several lectures, our goal is to introduce the notion of Quantum Error, and further, to understand that under what kind of conditions, Quantum Error Correction (QEC) is possible. To this end, we first need to explain how to describe general quantum errors mathematically, for which the basic issue is that quantum computers are **not** closed systems. In particular, to understand noise on our quantum computer, one needs to understand what happens to the states in the qubit memory when the computer and the environment undergo time evolution.

$$U : \left(\mathbb{C}^2\right)^{\otimes n} \otimes \mathscr{H} \to \left(\mathbb{C}^2\right)^{\otimes n} \otimes \mathscr{H},$$

where $\mathscr{H}$ is the Hilbert space of states for the cost of the universe. Not just being grandiose, it is a fact that current quantum computers have errors that come from cosmic rays per 10 seconds approximately. In some sense, it is safe to say that we will never be able to understand the U above. However, we can hope to understand the "reduced dynamics" of U when forgetting $\mathscr{H}$.

2 Quantum Errors

Definition (**Quantum operation**). *Let us consider a closed quantum system A with state space $\mathscr{H}_A$, a quantum operation (or quantum channel) on A, is whatever kind of thing we can get by coupling A to another system B, applying unitaries and making measurements to the composite system, and then forgetting the second system B.*

Remark 1. *Note that this definition here is informal and elusive, but here is enough to provide a high-level idea. Indeed, we will answer the following question in the next lecture: What is the mathematical characterization of quantum operations (quantum channels)?*

3 Mixed states

The motivation to introduce the notion of mixed states can be described as follow: If we perform a sequence of actions $\{\text{unitary}, \text{measurement}, \text{unitary}, \text{measurement}, \ldots\}$, it's annoying to book keep different potential outcomes since each measurement will collapse the state. The solution is basically to introduce mixed state by considering classical probability distributions on states. To this end, we will now call all the states in a Hilbert space "pure states".

Definition (Mixed states). *An* ensemble of (pure) states *on the Hilbert space* $\mathscr{H}$ *is defined to be any (finitely supported) probability distributions on pure states in* $\mathscr{H}$*, denoted by*

$$\{(|\psi_i\rangle \in \mathscr{H}, p_i)\}_{i\in\mathscr{I}},$$

where $0 \le p_i \le 1$ *for all* $i \in \mathscr{I}$ *satisfies* $\sum_{i\in\mathscr{I}} p_i = 1$ *and* $\mathscr{I}$ *is an index set. The* mixed state *determined by such an ensemble, is described by the associated* density operator*, which is defined as*

$$\rho := \sum_{i\in\mathscr{I}} p_i \frac{|\psi_i\rangle\langle\psi_i|}{\langle\psi_i|\psi_i\rangle}.$$

Example 2 (Ensemble of size one). *Every pure state* $|\psi\rangle$ *determines the mixed state* $\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$*. An interesting question immediately follows as: When is a mixed state actually a pure state?*

Example 3. *Let* $\mathscr{H} := \mathbb{C}^2$*, recall that measuring* $|+\rangle$ *in the computational basis will collapse into* $|0\rangle$ *with probability* $1/2$*, and* $|1\rangle$ *with probability* $1/2$*, according to the* Born rule*. In other words, the output of measurement in the computational basis can be described as the following mixed state (density operator):*

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|.$$

Remark 4. *An important observation is that different ensembles can yield equal density operators. Let us consider the following two ensembles:*

$$\mathscr{E}_1 := \left\{\left(|0\rangle, \frac{1}{2}\right), \left(|1\rangle, \frac{1}{2}\right)\right\} \text{ and } \mathscr{E}_2 := \left\{\left(|+\rangle, \frac{1}{2}\right), \left(|-\rangle, \frac{1}{2}\right)\right\}.$$

It turns out that they share the same density operator:

$$\begin{aligned}\rho(\mathscr{E}_2) =& \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \\ =& \frac{1}{2}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0| + \langle 1|}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0| - \langle 1|}{\sqrt{2}}\right) \\ =& \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \rho(\mathscr{E}_1).\end{aligned}$$

One may worry about that this sounds like a problem. However, it is actually not, because no measurements are able to distinguish between these two mixed states $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ *and* $\frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$*, that is, they should be considered the same mixed state. In a conclusion,* $\{$*mixed states in* $\mathscr{H}\}$ *has a 1-1 correspondence to* $\{$*density operators on* $\mathscr{H}\}$*, but does not have a 1-1 correspondence to* $\{$*ensembles in* $\mathscr{H}\}$*.*

Theorem 5. *Two ensembles* $\{(|\psi_i\rangle \in \mathscr{H}, p_i)\}_{i\in\mathscr{I}}$ *and* $\{(|\varphi_i\rangle \in \mathscr{H}, q_i)\}_{i\in\mathscr{I}}$ *share the same density operator if and only if there exists a unitary* $U := (u_{ij})_{i,j\in\mathscr{I}}$ *such that*

$$\sqrt{p_i}|\psi_i\rangle = \sum_{j\in\mathscr{I}} u_{ij}\sqrt{q_j}|\varphi_j\rangle.$$

Example 6. *A corollary following from Theorem* **??** *is that: All mixed states on a qubit are the convex hull of pure states on a qubit, which are all in the Bloch ball.*

According to the Theorem **??**, the time evolution of mixed states under a unitary U is then easy to present:

$$\begin{aligned}\mathscr{E}_1 = \{(|\psi_i\rangle \in \mathscr{H}, p_i)\}_{i\in\mathscr{I}} &\xrightarrow{U} \mathscr{E}_2 = \{(U|\psi_i\rangle \in \mathscr{H}, p_i)\}_{i\in\mathscr{I}}. \\ \rho(\mathscr{E}_1) = \sum_{i\in\mathscr{I}} p_i|\psi_i\rangle\langle\psi_i| &\xrightarrow{U} \rho(\mathscr{E}_2) = \sum_{i\in\mathscr{I}} p_i U|\psi_i\rangle\langle\psi_i|U^* = U\rho(\mathscr{E}_1)U^*.\end{aligned}$$

In other words, time evolution by a unitary U conjugates the mixed state. Let us then study the behavior of projective measurements on mixed states, which can now be computed likewise. We start with a warm-up by considering the case that we perform projective measurement of an observable $M = \sum \lambda P_\lambda$ on a pure state $|\psi_i\rangle$. The *Born rule* says that the probability of outcome λ given $|\psi_i\rangle$ is

$$p_i(\lambda) = \frac{\langle \psi_i | P_\lambda | \psi_i \rangle}{\langle \psi_i | \psi_i \rangle},$$

and this outcome implies that $|\psi_i\rangle$ has been collapsed into $P_\lambda |\psi_i\rangle$. In other words, projectively measuring with M turns $|\psi_i\rangle$ into the following mixed state

$$\rho_i := \sum_\lambda p_i(\lambda) \frac{(P_\lambda |\psi_i\rangle) \left(\langle \psi_i | P_\lambda^* \right)}{\langle \psi_i | P_\lambda^* P_\lambda | \psi_i \rangle}.$$

Since a projective operator P_λ satisfies $P_\lambda = P_\lambda^*$ and $P_\lambda^2 = P_\lambda$, we have

$$\rho_i := \sum_\lambda p_i(\lambda) \frac{(P_\lambda |\psi_i\rangle) \left(\langle \psi_i | P_\lambda^* \right)}{\langle \psi_i | P_\lambda^* P_\lambda | \psi_i \rangle} = \sum_\lambda \frac{P_\lambda |\psi_i\rangle \langle \psi_i | P_\lambda}{\langle \psi_i | \psi_i \rangle}.$$

Now suppose that we have an ensemble $\{(|\psi_i\rangle, q_i)\}_{i \in \mathscr{I}}$, i.e. a mixed state

$$\rho = \sum_{i \in \mathscr{I}} q_i \frac{|\psi_i\rangle \langle \psi_i|}{\langle \psi_i | \psi_i \rangle},$$

then projectively measuring with M on the mixed state ρ will yield a **mixture of mixed states** $\{\rho_i\}_{i \in \mathscr{I}}$ as follow

$$\sum_{i \in \mathscr{I}} q_i \rho_i = \sum_{i \in \mathscr{I}} q_i \sum_\lambda \frac{P_\lambda |\psi_i\rangle \langle \psi_i | P_\lambda}{\langle \psi_i | \psi_i \rangle} = \sum_\lambda \sum_{i \in \mathscr{I}} q_i \frac{P_\lambda |\psi_i\rangle \langle \psi_i | P_\lambda}{\langle \psi_i | \psi_i \rangle}$$

In particular, the probability that the output is λ is

$$\begin{aligned} p(\lambda) &= \sum_{i \in \mathscr{I}} q_i p_i(\lambda) \\ &= \sum_{i \in \mathscr{I}} q_i \frac{\langle \psi_i | P_\lambda | \psi_i \rangle}{\langle \psi_i | \psi_i \rangle} \\ &= \sum_{i \in \mathscr{I}} \frac{q_i}{\langle \psi_i | \psi_i \rangle} \operatorname{tr}(P_\lambda |\psi_i\rangle \langle \psi_i|) \\ &= \operatorname{tr}\left(P_\lambda \sum_{i \in \mathscr{I}} \frac{q_i |\psi_i\rangle \langle \psi_i|}{\langle \psi_i | \psi_i \rangle} \right) = \operatorname{tr}(P_\lambda \rho), \end{aligned}$$

and this outcome implies that the mixed state (with density operator) ρ has been "collapsed" into the mixed state (with density operator) $\frac{P_\lambda \rho P_\lambda}{\operatorname{tr}(P_\lambda \rho)}$.