CS 593/MA 592 - Intro to Quantum Computing
Spring 2024
Tuesday, April 23 - Lecture 15.1

Scribe: Wenbo [Note: not proofread by Eric]

Reading: Section 10.6 Nielsen and Chuang

1 Computing Rate and Distance of Toric Code

1.1 Toric Code

Let's start with a square $n \times n$ grid on a torus.

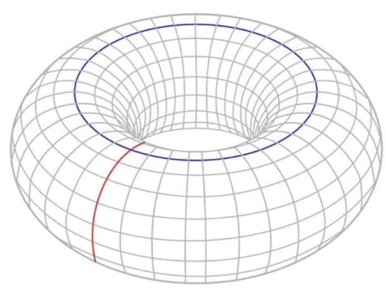

Figure 1: A torus

Rather than drawing the daunt-shaped figure shown in **??**, we cut it open to a square with periodic boundary conditions.

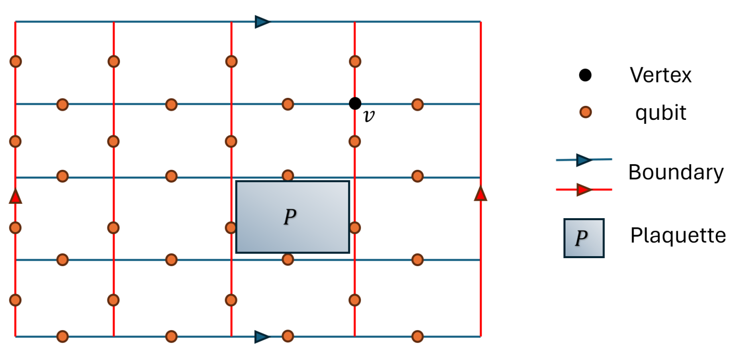

Figure 2: Cut a Torus open to a square with periodic boundary conditions. The lines with an arrow represent the cut

For each edge, we put a qubit on it. Therefore, there are a total of $2n^2$ qubits. For each vertex v, we label the 4 qubits around it with v_N, v_W, v_E, v_S like in **??**

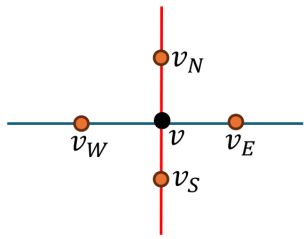

Figure 3: One way to label vertices around a vertex v

Then, we define a **vertex stabilizer** X_v for the vertex v as the following

$$X_v = X_{v_N} X_{v_E} X_{v_S} X_{v_W} \tag{1}$$

For each plaquette P, we label qubits around it similarly like the **??** below

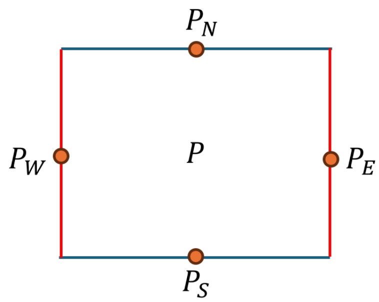

Figure 4: One way to label vertices around plaquette P

Again, we define a **plaquette stabilizer** Z_P for the plaquette P as the following

$$Z_P = Z_{P_N} Z_{P_E} Z_{P_S} Z_{P_W} \tag{2}$$

Definition 1.1 (Toric Code)**. Toric code** on $2n^2$ qubits is the stabilizer code C_n with stabilizer generators

$$S_n = \{X_v, Z_P | v \text{ vertex}, P \text{ plaquette in } n \times n \text{ grid on torus}\} \tag{3}$$

where X_v and Z_P are defined in **??** and **??**, respectively.

Example 1.2 (Toric Code)**.** *The* **??** *is a square* 4×4 *grid on a torus. Then,* X_v *for some vertex* v *and* Z_P *for some plaquette* P *can be visualized by the following diagram*

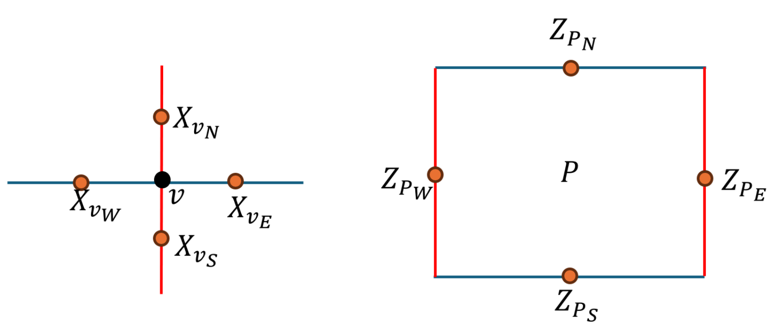

Figure 5: Z_P and X_v

1.2 Rate and Distance of Toric Code

Theorem 1.3. *Toric code (**??**) $C_n \leq (\mathbb{C}^2)^{\otimes(2n^2)}$ is a $[[2n^2, 2, n]]$ quantum error correction code.*

Definition 1.4 (String and Dual String)**.** A **string** γ is any connected path of edge in torus. A **dual string** $\widehat{\gamma}$ is any connected path in "dual" torus. (It is basically a path connecting some plaquettes)

Definition 1.5 (String Operators)**.** Let γ be a string defined in **??**. The **string operator** Z_γ is the Pauli operation formed by applying Z's along every qubit on the edges of γ.

Similarly, let $\widehat{\gamma}$ be a dual string. The **string operator** $X_{\widehat{\gamma}}$ is the Pauli operator formed by applying X's to qubits along the edges that $\widehat{\gamma}$ acrosses.

Lemma 1.6. *The string operators $X_{\widehat{\gamma}}$ and Z_γ generate the Pauli group G_{2n^2} up the phases*

Proof of **??**. Let $X_{\widehat{\gamma}}, Z_\gamma$ be string operators for strings of length 1. Clearly, these operators act on individual qubits on the torus. Therefore, they generate the group G_{2n^2}. □

Definition 1.7 (Closed String)**.** A string γ (or dual string $\widehat{\gamma}$) is **closed** if it gives a path without endpoints.

Lemma 1.8. *A string operators $X_{\widehat{\gamma}}$ or Z_γ commutes with every stabilizer generator in S_n (**??**) if and only if γ or $\widehat{\gamma}$ are closed.*

Proof of **??**. Consider $X_{\widehat{\gamma}}$. Suppose $\widehat{\gamma}$ is not closed. Then, there must exist different plaquettes $P_0 \neq P_1$ such that they are at the end of $\widehat{\gamma}$.

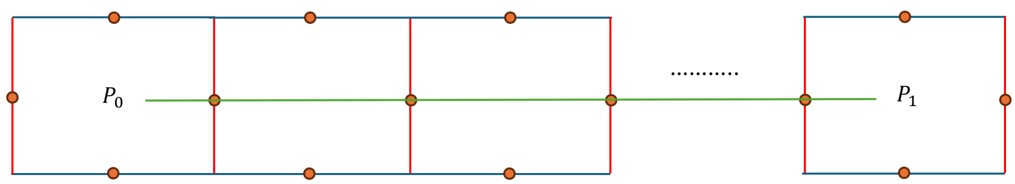

Figure 6: An example of $\widehat{\gamma}$ (green line)

Then, let's show $[X_{\widehat{\gamma}}, Z_{P_0}] \neq I$. For the sake of clarity, let's label the qubits in the following way

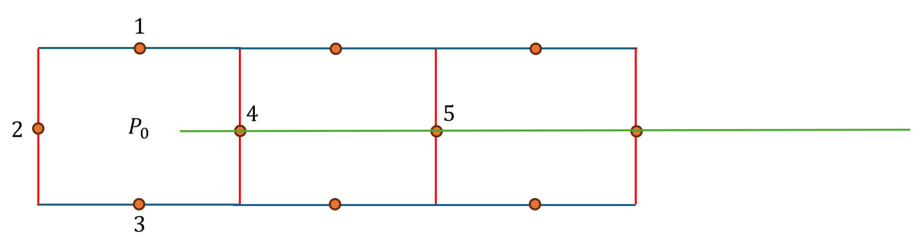

Figure 7: Labelled Qubits around the Plaquettes P_0

Then, by definition of the string operator and Z_{P_0}, we know

$$X_{\widehat{\gamma}} Z_{P_0} = \Bigg(\underbrace{\cdots}_{X\text{'s far away from } P_0} X_5 X_4 \Bigg) (Z_1 Z_2 Z_3 Z_4) \tag{4}$$

$$= X_4 (Z_1 Z_2 Z_3 Z_4) (\cdots X_5) \tag{5}$$

$$= -(Z_1 Z_2 Z_3 Z_4) X_4 (\cdots X_5) \tag{6}$$

$$= -Z_{P_0} X_{\widehat{\gamma}} \tag{7}$$

Similarly, for a string γ (a string in the torus, not a dual string). If γ is not closed, then Z_γ does not commute with the X_v's for the v's at the end of γ.

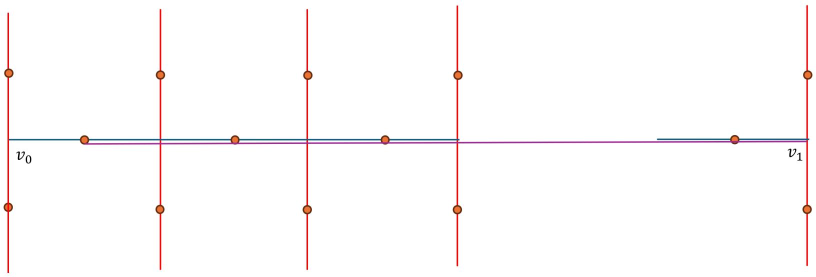

Figure 8: a γ string (purple line)

Conversely, suppose the string γ is closed.

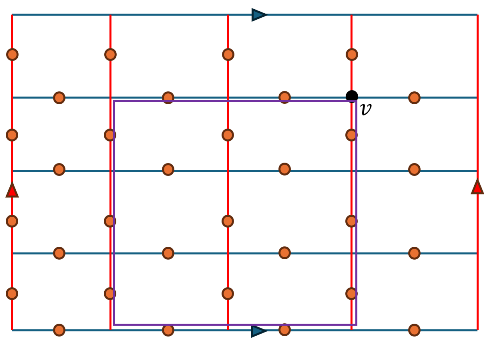

Figure 9: An example of closed string γ (Purple Line)

Need to show $[Z_\gamma, X_v] = [Z_\gamma, Z_P] = I$. $[Z_\gamma, Z_P] = I$ is trivial. So, it suffices to show $[Z_\gamma, X_v] = I$. Again, let's label the qubits around some vertex v on the path γ as the following

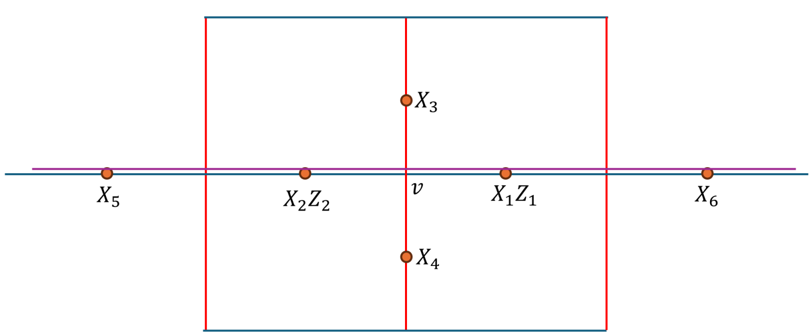

Figure 10: Labelled Qubits around the Vertex v

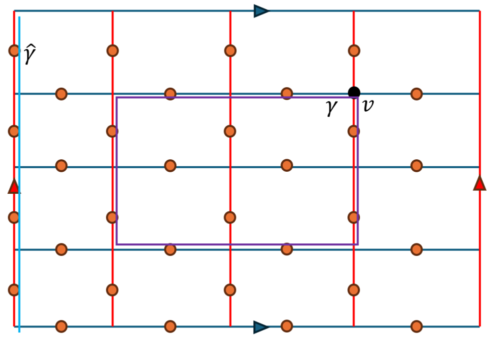

Figure 11: γ is a closed string, and it is a boundary of the union of four plaquettes in the middle. $\widehat{\gamma}$ is a closed string, but it is not a boundary of any union of plaquettes

Then, again, by the definition of the string operator and X_v, we know

$$Z_\gamma X_v = \left(\underbrace{\cdots}_{Z\text{'s far away from } v} Z_6 Z_5 Z_2 Z_1 \right)(X_1X_2X_3X_4) \tag{8}$$

$$=(Z_2Z_1)(X_1X_2X_3X_4)(\cdots Z_6Z_5) \tag{9}$$

$$=-Z_2(X_1X_2X_3X_4)(\cdots Z_6Z_5Z_1) \tag{10}$$

$$=(-1)^2(X_1X_2X_3X_4)(\cdots Z_6Z_5Z_1Z_2) \tag{11}$$

$$=X_vZ_\gamma \tag{12}$$

We can use similar arguments for closed dual string $\widehat{\gamma}$ □

Lemma 1.9. *An element $g \in G_{2n^2}$ commutes with every element of S_n if and only if g is a composition of closed string operators. In other words, closed string operators generate the group of logical Pauli operators on the toric code C_n, where **logical Pauli operators** are the Paulis commute with all stabilizers*

Sketched Proof of **??**. The "if" direction directly follows the previous lemma.

To complete the proof, we only need to check the "only if" direction. First of all, one needs to check "if g does not commute with some plaqutte Z_{P_0}, then there exists a different plaquette $P_1 \neq P_0$ such that g does not commute with Z_{P_1} either."

Similarly, if g does not commute with some X_{v_0}, then there must exist some $v_1 \neq v_0$ such that g does not commute with X_{v_1} either.

Using these facts, one can show g is not a composition of closed string operators. □

Remember, our goal is to show $\dim C_n = 4 = 2^2$ and $\mathrm{dist}(C_n) = n$. To show $\dim C_n = 4 = 2^2$, it suffices to show

$$\pi(\langle S_n \rangle)^\perp / \pi(\langle S_n \rangle) \simeq \mathbb{F}_2^2 \oplus \mathbb{F}_2^2 \tag{13}$$

Definition 1.10 (Boundary)**.** A string γ is a **boundary** if it is closed and can be identified as the boundary of the union of some plaquettes.

Example 1.11 (**??**)**.** *Let's consider the following two closed strings γ, γ' (purple line and light blue line, respectively)*

Lemma 1.12. *A closed string operation Z_γ acts non-trivially on C_n if and only if γ is not a boundary*

Theorem 1.13. *A group of non-trivial Pauli logical errors is generated by the following 4 string operations*

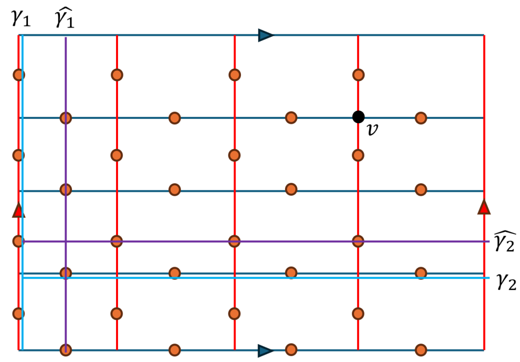

Figure 12: γ_1, γ_2 are some closed strings (light blue lines). $\widehat{\gamma}_1, \widehat{\gamma}_2$ are some closed dual strings (purple lines)

Corollary 1.14. *The smallest non-trivial Pauli logical error on C_n requires us to apply a cycle of X's or Z's that wind all the way around the torus. The smallest such thing involves n qubits. Therefore, the distance of the code is n. In particular, C_n encodes 2 logical qubits.*

Can we generalize toric code to encode more logical qubits? The answer is yes, but we need a higher genus surface. Here are two examples

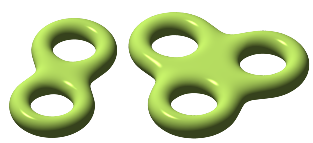

Figure 13: Double Torus and Triple Torus. They encode 4 and 6 logical qubits, respectively. In general, *n*-ple torus encodes 2*n* logical qubits