CS 593/MA 592 - Intro to Quantum Computing
Spring 2024
Thursday, March 5 - Lecture 9.1

Today's scribe: Jesse [Note: note proofread by Eric]

Reading:

1. 5.2 of [NC]
2. 14 of [Kitaev-Shen-Vynlyi]

Agenda:

1. What quantum simulation does not accomplish
2. Phase estimation
3. QMA

1 Recap of Quantum Simulation

Last time, given both a state $|\psi\rangle \in (C^2)^{\otimes n}$ and a Hamiltonian H on $(C^2)^{\otimes n}$, we can simulate the time evolution of the Hamiltonian e^{-iHt} on $|\psi\rangle$. This is a source of quantum advantage. But this does not mean that quantum simulation can find important states. Arguably the most important type of question in "applied" quantum mechanics is the following.

> Given a local Hamiltonian H, for some system discretized on some qubits, find a lowest energy eigenvector of H.

This question is hard to answer in general, even for "physically reasonable" H and even if we have a quantum computer.

Example: Last time we discussed the Heisenberg model.

$$H = -\sum_{i=1}^{M-1} J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1} \tag{1}$$

A variation of the Heisenberg model is the Ising model where

$$J_X = J_Y = 0, J_Z \text{ depends on } i \tag{2}$$

This is a classical statistical mechanical system because H is already diagonal in the computational basis. We can set up an Ising model on any edge weighted graph $(V, E, J : E \to \mathbb{R})$. This determines a Hamiltonian by putting a qubit at every vertex and defining

$$H = -\sum_{e=\{i,v\}\in E} J(e) Z_i Z_j \tag{3}$$

It turns out that finding the lowest energy states of these generalized Ising models is equivalent to the weighted max-cut problem which is NP-hard. If we have a complicated Hamiltonian, then it is QMA-hard.

2 Phase Estimation

Given a eigenvector $|u\rangle$ of a Hamiltonian H on n qubits, can we estimate its energy spectrum? Equivalently, we can find the eigenvalue θ such that $H|u\rangle = \theta|u\rangle$. **Classically** It is not clear how to do this with less than exponential space unless H and $|u\rangle$ are sparse in the computational basis. Because we are performing a matrix vector multiply to find the scalar multiple of the eigenvector $|u\rangle$, which has exponential time complexity in the number of qubits n. **Quantumly** There exists a good approximation algorithm (assuming H is "sensible", $||H||$ is not too big)

Remark. *If we evolve the system for one unit of time then it is equivalent to multiplying $|u\rangle$ by an exponentiated phase.*

$$H|u\rangle = \theta|u\rangle \implies e^{2\pi i \frac{H}{2\|H\|}}|u\rangle = e^{2\pi i \frac{\theta}{2\|H\|}}|u\rangle \tag{4}$$

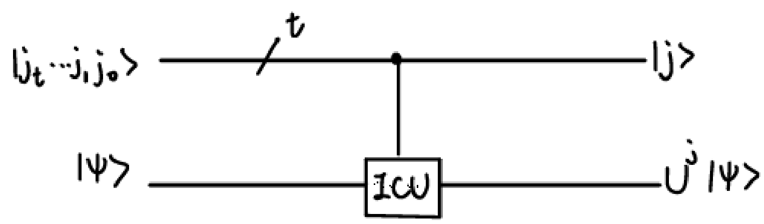
If we have $\|H\| \leq poly(O(n))$, then we can apply the inverse quantum Fourier transform and get back the phase θ roughly.

So we can reduce to the following problem.

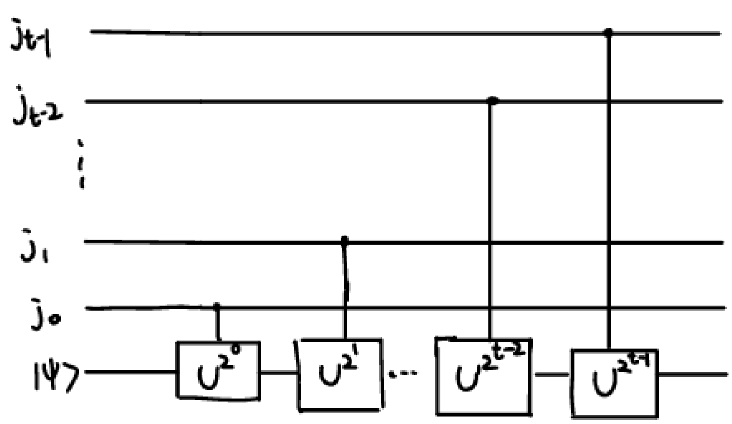
Given a unitary U and an eigenvector $|u\rangle$ with

$$U|u\rangle = e^{2\pi i\theta}|u\rangle \tag{5}$$

Find an approximation to the phase θ. We will assume we are able to implement $C-U^{2^k}$ for arbitrary k via oracle. Using the controlled U we can build a unitary called "$t-ICU$" t-bit integer controlled-U.



This is a unitary operation that is controlled by t ancillas, where U is applied to $|\psi\rangle$ when an ancilla qubit in $|1\rangle$ state, e.g. if $|j\rangle = |1\ldots 11\rangle = 2^t - 1$. We build this "t-ICU" operator from controlled-U gates as follows, where the evolution time of $|\psi\rangle$ is controlled by the binary representation of the state of the ancilla qubits $|j\rangle$.



Remark.

$$(t-ICU)|j\rangle \otimes |\psi\rangle = |j\rangle \otimes U^j|\psi\rangle$$

if $|\psi\rangle = |u\rangle$ *the eigenstate of* H, *then*

$$(t-ICU)|j\rangle \otimes |u\rangle = e^{2\pi i\theta}|j\rangle \otimes |u\rangle$$

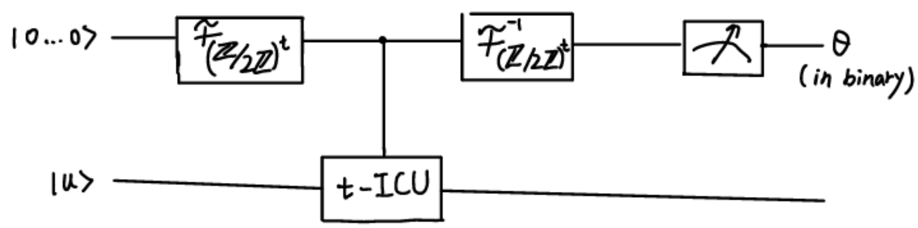
As usual, we might guess that it would be a good idea to apply $t-ICU$ to a uniform superposition state on t qubits, $\frac{1}{2^{t/2}}\sum_{j=0}^{2^t-1}|j\rangle\otimes|u\rangle$. This yields the following representation, where we get all the different powers of the eigenvalue $e^{2\pi i\theta}$ from $j=0$ to $j=2^t-1$,

$$\begin{aligned}(t-ICU)\frac{1}{2^{t/2}}\sum_{j=0}^{2^t-1}|j\rangle\otimes|u\rangle &= \frac{1}{2^{t/2}}\sum_{j=0}^{2^t-1}e^{2\pi ij\theta}|j\rangle\otimes|u\rangle \\ &= \frac{1}{2^{t/2}}(|0\rangle+e^{2\pi i2^{t-1}\theta}|1\rangle)(|0\rangle+e^{2\pi i2^{t-2}\theta}|1\rangle)\dots(|0\rangle+e^{2\pi i2^{0}\theta}|1\rangle)\otimes|u\rangle\end{aligned}$$

This looks suspiciously familiar to QFT. More precisely, if $0\leq\theta<1$ has an exact representation as a binary fraction using t bits, i.e. $\theta=0.\theta_1\theta_2\dots\theta_t$, then what we see is precisely the Fourier transform of this computational basis vector

$$\mathscr{F}_{(\mathscr{Z}/2\mathscr{Z})^t}|\theta_n\theta_{n-1}\dots\theta_2\theta_1\rangle \tag{6}$$

So we could apply the inverse Fourier transform. Thus at least if $\theta=0.\theta_1\theta_2\dots\theta_t$ the following circuit does the job of finding θ given $|u\rangle$.



Remark.

$$H^{\otimes t}|0\dots0\rangle=(\mathscr{F}_{\mathscr{Z}/2\mathscr{Z}})^{\otimes t}=\mathscr{F}_{(\mathscr{Z}/2\mathscr{Z})^t}|0\dots0\rangle \tag{7}$$

What if θ is not a t-bit binary fraction? The procedure still works, but does not succeed perfectly, with some probability.

3 QFT protocol summary

Input:

1. $n,\varepsilon>0, t=n+\lceil\log(2+1/\varepsilon)\rceil$
2. oracle access to $t-ICU$'s (or alternatively, $C-U^{2^k}$)
3. an eigenstate $|u\rangle$ of U with $U|u\rangle=e^{2\pi i\theta}|u\rangle$

Output:
(Best) n bit approximation to θ in binary

Complexity:

1. $O(t^2)$ operations
2. One call to $t-ICU$
3. Success with prob $1-\varepsilon$

We can trade-off the number of bit precision n with the probability of success $1-\varepsilon$. What if $|u\rangle$ is not an eigenstate? but an arbitrary state $|\psi\rangle$ is given. In principle, we do not know the amplitude ahead of time. We can write $|\psi\rangle$ as a superposition of eigenstates $|u\rangle$ of U, where $U|u\rangle = e^{2\pi i\theta_u}|u\rangle$. We can sample from the eigenbasis of U.

$$|\psi\rangle = \sum_{u:\text{eigenvectors of } U} z_u|u\rangle \tag{8}$$

If we input $|\psi\rangle$ to QPE then after the measurement, the output will be an approximation to θ_u with probability (HW5)

$$|z_u|^2(1-\varepsilon) \tag{9}$$