CS 593/MA 595 - Intro to Quantum Computation
Theoretical Homework 1

Due Wednesday, September 3 at 11:59PM (upload to Brightspace)

THW will be a little longer the first few weeks of the semester. We need to get a solid grounding in linear algebra and notation before can get to the good stuff!

Recommended exercises from Mike and Ike (not to be turned in): 2.2, 2.7, 2.10, 2.20, 2.26, 2.29-2.33, 2.42, 2.44, 2.45, 2.46 and 2.47.

1. This first exercise consists of a few drills.
   (a) Recall that $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Determine what the Pauli X, Y and Z operators looks like in the $|+\rangle, |-\rangle$ basis. Do the same for the scalar matrices $\pm iI = \begin{pmatrix} \pm i & 0 \\ 0 & \pm i \end{pmatrix}$.
   (b) Let z_j $(j = 0, \ldots, 7)$ be the amplitudes of $|+\rangle^{\otimes 3} \in \left(\mathbb{C}^2\right)^{\otimes 3}$ in the computational basis. Compute them.

2. We now really dig into the precise relationship between bras and kets.

   Let $\mathcal{H}$ be a finite dimensional Hilbert space. We define the *dual* Hilbert space $\mathcal{H}^*$ to be the set of all linear transformations $\rho : \mathcal{H} \to \mathbb{C}$, that is:

$$\mathcal{H}^* := \{\rho : \mathcal{H} \to \mathbb{C} \mid \rho \text{ is a linear function}\}.$$

   Recall that if $|\psi\rangle \in \mathcal{H}$, then $\langle\psi|$ is the linear transformtation

$$\begin{aligned} \langle\psi| : \mathcal{H} &\to \mathbb{C} \\ |\phi\rangle &\mapsto \langle\psi \mid \phi\rangle \end{aligned}$$

   where $\langle\psi \mid \phi\rangle$ is the inner product.

   (a) Show that $\mathcal{H}^*$ is a vector space with the same dimension as $\mathcal{H}$.
   (b) Show that the function

$$\begin{aligned} F : \mathcal{H} &\to \mathcal{H}^* \\ |\psi\rangle &\to \langle\psi| \end{aligned}$$

   is a bijection. **Warning:** F is NOT linear (it is *anti-linear* or *conjugate-linear*), so you more-or-less need to show injectivity and surjectivity directly. For surjectivity, use the part (a) (in particular, the assumption that $\mathcal{H}$ is finite dimensional).

   In fact, we won't do this, but it's possible to define an inner product structure on $\mathcal{H}^*$ so that the map $|\psi\rangle \mapsto \langle\psi|$ becomes an anti-linear isometry. The moral of the story is that Hilbert spaces are ALMOST isometrically isomorphic to their dual spaces—the only finnicky thing is that the "isomorphism" is not linear, it's anti-linear! This is called the Riesz representation theorem. It's true more generally, e.g. for infinite dimensional Hilbert spaces too (except in this case $\mathcal{B}(\mathcal{H})$ is defined to consist only of "bounded" operators–this explains the $\mathcal{B}$ in the notation–and the usefulness of the Riesz representation theorem is why we define Hilbert spaces to be *complete*).

3. A big part of the power of bra-ket notation is explained by the following exercise.

   Let $\mathcal{H}$ be a finite-dimensional Hilbert space and define $\mathcal{B}(\mathcal{H})$ to be the set of all linear transformations $A : \mathcal{H} \to \mathcal{H}$.

   (a) Show that $\mathcal{B}(\mathcal{H})$ is a vector space. What is its dimension?

   (b) Show that the map
   $$\begin{aligned} \mathcal{H} \otimes \mathcal{H}^* &\to \mathcal{B}(H) \\ |\phi\rangle \otimes \langle\psi| &\mapsto |\phi\rangle\langle\psi| \end{aligned}$$
   is a vector space isomorphism. Here $|\phi\rangle\langle\psi|$ is the linear operator
   $$\begin{aligned} |\phi\rangle\langle\psi| : \mathcal{H} &\to \mathcal{H} \\ |\rho\rangle &\mapsto |\phi\rangle\langle\psi \mid \rho\rangle = \langle\psi \mid \rho\rangle|\phi\rangle \end{aligned}$$
   and we are using the fact that to define a linear map of the form $U \otimes V \to W$ (where U, V, W are vector spaces), it suffices to specify what it looks like on separable vectors (because separable vectors in $U \otimes V$ are a spanning set).

4. Let's unravel the spectral theorem in some important special cases.

   (a) Show that a normal operator is Hermitian if and only if all of its eigenvalues are real.

   (b) Show that a normal operator is unitary if and only if all of its eigenvalues are complex numbers with modulus 1.

   (c) Differing slightly from the book, for the moment let us *define* an orthogonal projector to be a linear operator $P : \mathcal{H} \to \mathcal{H}$ such that $P^2 = P$ and $P^\dagger = P$. Show that P is an orthogonal projector if and only if P is unitarily diagonalizable with all eigenvalues equal to either 0 or 1.

5. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two (finite-dimensional) Hilbert spaces, let $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ be two linear operators, and let $\sum_{j=0}^{m} w_j|a_j\rangle \in \mathcal{H}_1$, $\sum_{k=0}^{n} v_k|b_k\rangle \in \mathcal{H}_2$ be two arbitrary superpositions. (That is, the kets $|a_j\rangle$ and $|b_k\rangle$ are arbitrary; in particular, they need not be computational basis vectors.)

   Show
   $$(A \otimes B)\left[\left(\sum_{j=0}^{m} w_j|a_j\rangle\right) \otimes \left(\sum_{k=0}^{n} v_k|b_k\rangle\right)\right] = \left(\sum_{j=0}^{m} w_j A|a_j\rangle\right) \otimes \left(\sum_{k=0}^{n} v_k B|b_k\rangle\right)$$