Meeting 1.2

I. Triangulations of (compact, PL) manifolds

II. Basic questions, and why dimension 3 is special

III. Time permitting: other encodings of 3-manifolds, knots & links

Next time: complexity theory

I. Triangulations

Recall our claim from last class: every (compact) PL d-manifold is PL-homeomorphic to a (finite) d-dimensional simplicial complex s.t. the link of every vertex is a PL d-1 sphere.

We will call such a simplicial complex a triangulation (of a manifold).

Link of vertex v is the union of all simplices τ such that τ and v share a simplex, but τ and v are disjoint.

PL Homeomorphism

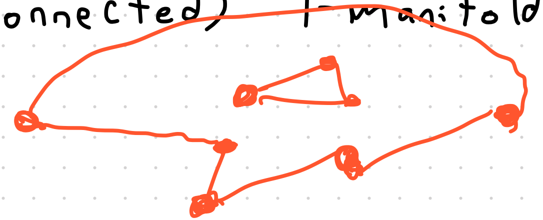
A homeomorphism $F: M \to N$ is a PL homeomorphism, if in all coordinate charts (of the PL structures of M and N), F is a PL homeo b/w open subsets of $\mathbb{R}^n$.

Two triangulations are combinatorially equivalent if they have isomorphic refinements.

e.g.

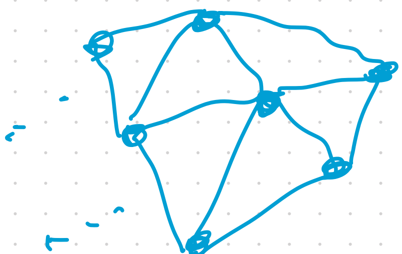
Let $\mathcal{T}_i$ be a triangulation of M_i, $i=1,2$.
Then M_1 and M_2 are PL homeomorphic if and only if $\mathcal{T}_1$ and $\mathcal{T}_2$ are combinatorially equivalent.

Example Every 2-regular graph is a triangulation of a (possibly disconnected) 1-manifold.

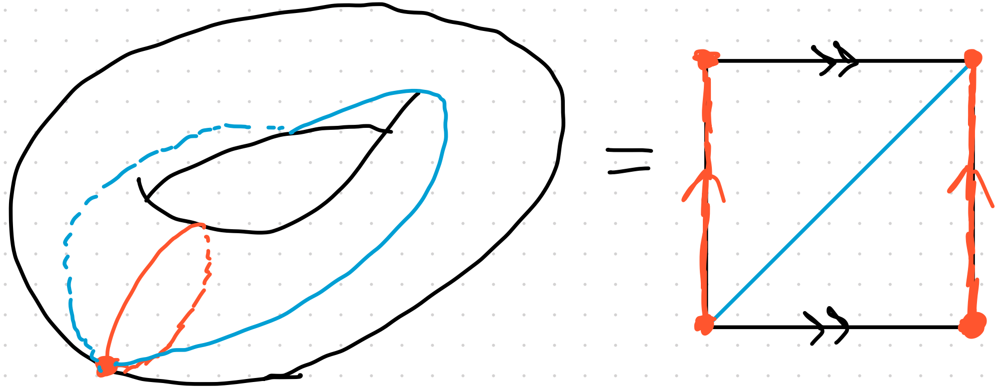


$\cong$ two copies of S^1.

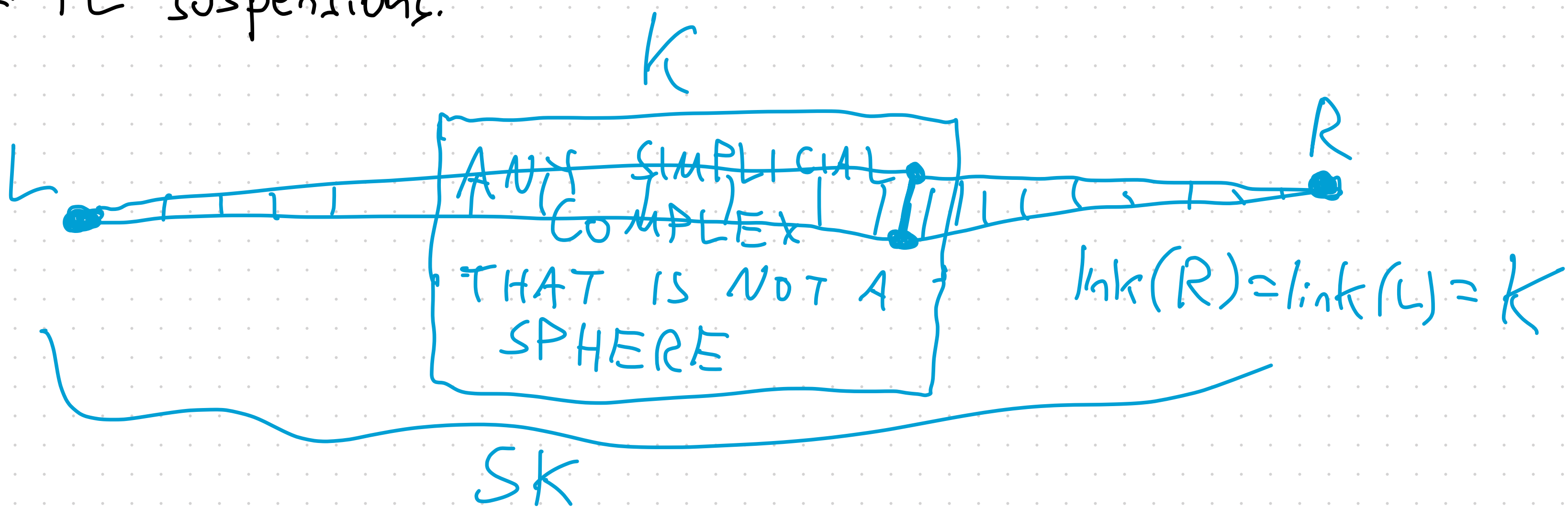
Example Every 2-dim simplicial complex where each edge is contained in exactly 2 triangles is a triangulation of a surface ("surface" = "2-dimensional manifold").



Non-examples 1.



2. PL suspensions.



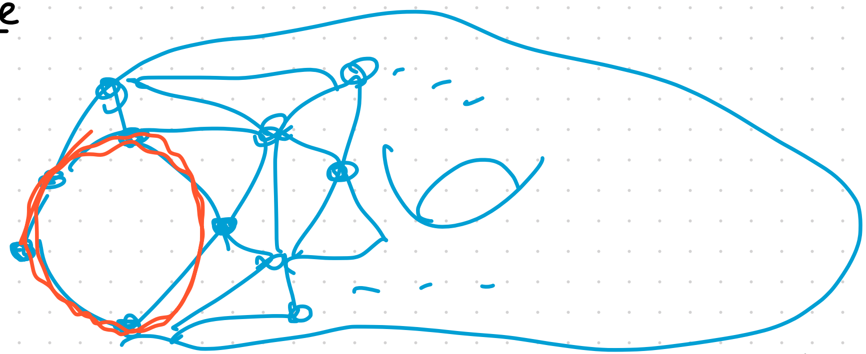
<u>Warning</u> A simplicial complex may be homeomorphic (but not PL homeomorphic) to a manifold, even if the complex is not what we're calling a triangulation.

<u>Double suspension theorem</u> If M is any d-manifold that has the same homology as S^d, then the double suspension of M, $S^2 M$, is a topological $d+2$ sphere.

Manifolds with boundary

In defn of manifold, just replace $\mathbb{R}^d$ with $\mathbb{R}^{d-1} \times [0,\infty)$.

Example



triangulation of torus w/
one boundary component

For triangulations of manifolds w/ boundary, the links of boundary points should be $d-1$ disks.

Standing implicit assumptions

Abuse of notation: "manifold" will often mean "triangulation of a (closed, compact, orientable) manifold."

But sometimes not.

If unclear, please ask!

II. Basic questions, and why $d=3$ is the best (to me)

If we want to use triangulations of manifolds as input to computer programs designed to calculate properties of manifolds, at the very least, we would like to recognize when a simplicial complex is a valid triangulation.

How would we do this?

Work recursively and "down" from d all the way to 0.
Pick a vertex v and calculate $\text{link}(v)$.
Then determine if $\text{link}(v)$ is a $(d-1)$-dimensional triangulation.
If not, stop. If yes, then decide if $\text{link}(v)$ is a $d-1$ sphere. If yes, move to next vertex. Repeat.

The curse of uncomputability

Given a d-manifold, is it a d-sphere?

Given simplicial complex, is it a triangulation?

d	PL d-sphere recognition	$(d+1)$-dimensional triangulation recognition	d-manifold homeo.
0	easy	easy	easy
1	easy	easy	easy
2	easy*	easy	easy*
3	$NP \cap coNP^{t}$	algorithmically possible but not easy	algorithmic, but complexity unknown
4	NO IDEA!!	NO IDEA!!	
5	NOT POSSIBLE!!		
6			
⋮			

(*): First decide if 2-manifold; then compute H_* using SNF on d, cellular boundary map.

(t): will discuss next week

Other nice things about 3-manifolds

Moise's Theorem In dimension 3

$$\text{TOP} = \text{PL} = \text{DIFF}.$$

Poincaré conjecture true

If M is has homotopy groups of S^3, then $M \cong S^3$.

$\Uparrow$

Geometrization!

d	
0	$T = P = D$
1	
2	
3	$T = P = D$
4	$T \subsetneq P = D$
5	$\vdots$
6	$\vdots$
7	$T \subsetneq P \subsetneq D$

exotic spheres...