Meeting 10.1: From TQFT to TQC, a brief history

I. Atiyah + Witten

II. Reshetikhin - Turaev + Turaev

III. Turaev - Viro + Barrett-Westbury

IV. Kitaev + Freedman

V. Freedman - Kitaev - Larsen - Wang

VI. Levin - Wen

Kitaev's motivation for introducing toric code (and generalizations to other finite groups I will mention later) was to address fault tolerance problem USING HARDWARE.

He doesn't use language of TQFT directly, but was clearly inspired by it, since anyons were understood to be the "particles" that can arise in certain exotic (topological) QFTs.

I. Atiyah + Witten

1988 – Atiyah defines topological quantum field theory.

Mathematically rigorous!

Uses language of cobordisms and functors.

Inspired by work (esp. of Witten) on (general, not-entirely-rigorous) supersymmetric quantum field theory, and Segal's axioms for conformal field theory...

TOPOLOGICAL QUANTUM FIELD THEORIES

by Michael ATIYAH

To René Thom on his 65th birthday.

1. Introduction

In recent years there has been a remarkable renaissance in the relation between Geometry and Physics. This relation involves the most advanced and sophisticated ideas on each side and appears to be extremely deep. The traditional links between the two subjects, as embodied for example in Einstein's Theory of General Relativity or in Maxwell's Equations for Electro-Magnetism are concerned essentially with classical fields of force, governed by differential equations, and their geometrical interpretation. The new feature of present developments is that links are being established between *quantum* physics and *topology*. It is no longer the purely *local* aspects that are involved but their *global* counterparts. In a very general sense this should not be too surprising. Both quantum theory and topology are characterized by discrete phenomena emerging from a continuous background. However, the realization that this vague philosophical view-point could be translated into reasonably precise and significant mathematical statements is mainly due to the efforts of Edward Witten who, in a variety of directions, has shown the insight that can be derived by examining the topological aspects of quantum field theories.

The best starting point is undoubtedly Witten's paper [11] where he explained the geometric meaning of super-symmetry. It is well-known that the quantum Hamiltonian corresponding to a classical particle moving on a Riemannian manifold is just

Pub. IHES (1988)

TQFT in a nutshell

$\Bbbk$: a field (or other unital commutative ring...)

$Cob(d)$: d-dimensional oriented cobordism category

Objects$(Cob(d))$: oriented, smooth, closed d-manifolds

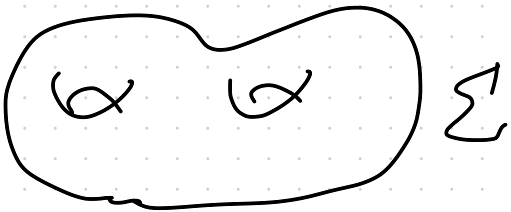
Mor$(Cob(d))$: oriented, smooth $(d+1)$-manifolds M, w/ $\partial M = \Sigma_0 \sqcup \Sigma_1$. M is a morphism $M: \Sigma_0 \to \Sigma_1$

$\otimes$: disjoint union

A $(d+1)$-dimensional TQFT is a "$\otimes$-respecting linearization of $Cob(d)$," i.e. a $\otimes$-functor $Z: Cob(d) \to Vec(\Bbbk)$

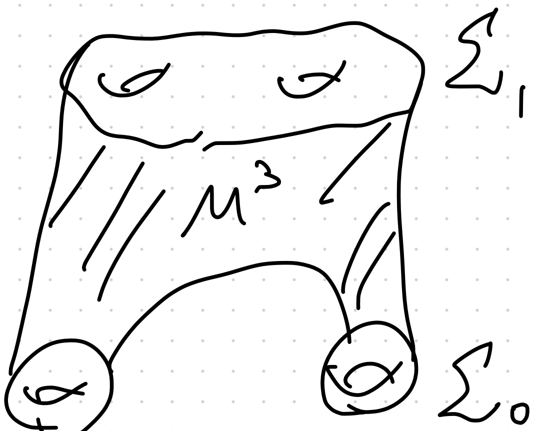
might as well assume finite dim.

Schematic $\quad d=2,\ \Bbbk=\mathbb{C}$



$\longmapsto Z(\Sigma) \leftarrow$ F.d. vector space over $\mathbb{C}$

$\longmapsto Z(\Sigma_1 \sqcup \Sigma_2) \cong Z(\Sigma_1) \otimes Z(\Sigma_2)$



$\longmapsto Z(M): Z(\Sigma_0) \to Z(\Sigma_1)$

linear map

Hermitian and unitary TQFT

If $\mathbb{k} = \mathbb{C}$, we can ask

$$Z(-M) = Z(M)^*$$

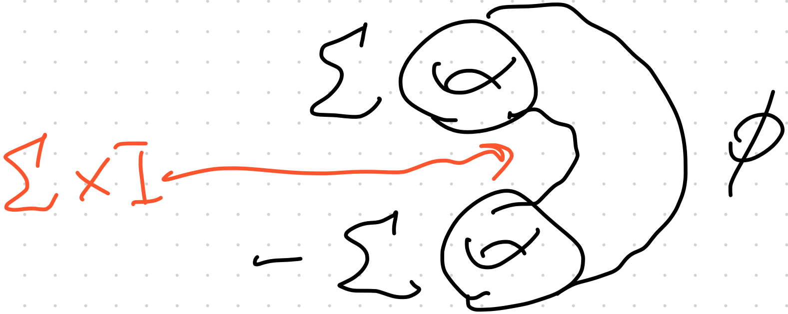
adjoint.

M w/ reversed orientation and swapped boundary pieces

For all M. If this holds, call the TQFT <u>hermitian</u>

It is <u>unitary</u> if moreover, the pairing

$$\phi \mapsto$$



$$Z(\Sigma\times I): Z(\Sigma)\otimes Z(-\Sigma)\longrightarrow Z(\phi)=\mathbb{C}$$

$$Z(-\Sigma) = Z(\Sigma)^*$$

dual vector space

If this pairing is positive definite and Z is Hermitian, then we say Z is unitary.

If Z is unitary, then $Z(\Sigma)$ is a Hilbert space.

<u>E.g.</u> (Toric code)

$$Z(\Sigma) := \operatorname{span}_{\mathbb{C}} H_1(\Sigma; \mathbb{Z}/2) \quad \text{if } \Sigma \text{ connected}$$

$$Z(\Sigma_1 \sqcup \Sigma_2) := Z(\Sigma_1) \otimes Z(\Sigma_2)$$

If $\partial M = \Sigma_0 \sqcup \Sigma_1$, then

$$Z(M) : Z(\Sigma_0) \longrightarrow Z(\Sigma_1)$$

linearizes the correspondence

$$M^* \subseteq H_1(\Sigma_0; \mathbb{Z}/2) \times H_1(\Sigma_1; \mathbb{Z}/2)$$

$$M^* = \{ (\alpha, \beta) \mid [\alpha] = [\beta] \text{ in } H_1(M; \mathbb{Z}/2) \}$$

Also in 1988, Atiyah asked

Is there an intrinsically 3-dimensional explanation for why Jones polynomial is an invariant of knots?

Jones had discovered it in 1984. Understood only diagrammatically at that time, e.g. as a normalization of Kauffman bracket

$$\langle \bigcirc \rangle = -q^{1/2} - q^{-1/2}$$

$$\langle \times \rangle = -q^{1/4} \langle)\,(\rangle - q^{-1/4} \langle \asymp \rangle$$

1989 - Witten argues (not 100% rigorously) that for q a root of unity, the Kauffman bracket can be used to build a (2+1)-dim TQFT.

Based on quantizing Chern-Simons theory w/ gauge group $G = SU(2)$. Different roots of unity yield different TQFTs.

Commun. Math. Phys. 121, 351–399 (1989)

Communications in Mathematical Physics

Quantum Field Theory and the Jones Polynomial *

Edward Witten **

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Abstract. It is shown that $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in $1+1$ dimensions.

II. Reshetikhin - Turaev + Turaev

Witten's construction not rigorous. Eventually made rigorous, but in the meantime Reshetikhin and Turaev did give a mathematically rigorous construction using quasi-triangular Hopf algebras and diagrammatic (or skein) constructions of TQFTs.

The Witten-Reshetikhin-Turaev uses the category of finite dimensional representations of a quasi-triangular Hopf algebra. When one uses $U_q sl_2$, one recovers the Jones-Kauffman TQFT for that specific q.

Turaev generalized further to arbitrary

Modular Tensor Categories.

(If H is q.tri. Hopf algebra, then $\mathrm{Rep}(H)$ is a modular tensor category.)

It turns out, once-extended $(2+1)$-dimensional TQFTs are entirely determined by a modular tensor category (w/ one additional small choice)

Recentish theorem of Douglas, Schommer-Pries, Vicary, et al...

A once-extended TQFT is a "usual" $(d+1)$-dimensional TQFT that also associates data to every $(d-1)$-manifold in functorial way...

Making this precise involves "higher tensor categories"

E.g.

$$M^3 \longmapsto Z(M) : Z(\partial M_0) \to Z(\partial M_1)$$

linear map

$$\Sigma \longmapsto \text{vector space } Z(\Sigma)$$

$$S^1 \longmapsto \text{category } Z(S^1)$$

it is a modular tensor category...

One can study even further extended TQFTs...

e.g. Fully-extended TQFT

$d+1$ manifold $\rightsquigarrow$ linear map

d manifold $\rightsquigarrow$ vector space

$d-1$ manifold $\rightsquigarrow$ category

$d-2$ manifold $\rightsquigarrow$ 2-category

$\vdots$

point $\rightsquigarrow$ d-category

cf. Baez-Dolan cobordism hypothesis proved by Lurie.

III. Turaev-Viro + Barrett-Westbury

Turaev-Viro showed (1993?) how to construct a fully extended 3-d TQFT from a modular tensor category.

Don't get anything that Reshetikhin-Turaev construction doesn't already provide.

Barrett-Westbury defined <u>spherical tensor categories</u>, and showed Turaev-Viro "works" for any spherical tensor category

Turaev-Viro + Barrett-Westbury:

$\bullet \mapsto$ spherical tensor category $\mathcal{C}$

$S^1 \mapsto$ Drinfel'd center $Z(\mathcal{C})$

(always a modular tensor category)

$\Sigma \mapsto$ Vector space $Z(\Sigma)$

(agrees R-T construction for $Z(\mathcal{C})$)

Eg.

$$\mathcal{C} = G\text{-Vec}$$

the category of G-graded finite dimensional vector spaces over $\mathbb{C}$.

Object in $\mathcal{C}$ looks like

$$V = \bigoplus_{g \in G} V_g$$

where V_g is a f.d. vect. space.

Morphism

$$F: V = \bigoplus_{g \in G} V_g \longrightarrow W = \bigoplus_{h \in G} W_h$$

is a sum of linear maps

$$F_g: V_g \longrightarrow W_g,$$

i.e. $$F = \bigoplus_{g \in G} F_g.$$

$$V \otimes W = \left(\bigoplus_g V_g\right) \otimes \left(\bigoplus_h W_h\right)$$

$$= \bigoplus_x (V \otimes W)_x$$

$$(V \otimes W)_x = \bigoplus_{\substack{g,h \\ gh = x}} V_g \otimes W_h$$

$$= \bigoplus_g V_g \otimes W_{g^{-1}x}$$

Kitaev's paper, esp. 2^{nd} half, is essentially building the Turaev-Viro-Barrett-Westbury TQFT associated to G-Vec.

Toric code is special case $G = \mathbb{Z}/2$.