Meeting 10.2: Topological quantum computing, I

I. Anyons from elementary excitations in toric code

II. TQC and TQFT go hand in hand.

I. Anyons from elementary excitations in toric code

Given a cellulation w/ N edges of a genus g surface S_g, toric code yields a 4^g-dimensional code space

$$\mathcal{H} \subseteq (\mathbb{C}^2)^{\otimes N}$$

$$\mathcal{H} = \{ |\psi\rangle \mid X_v|\psi\rangle = |\psi\rangle = Z_p|\psi\rangle \quad \forall \text{ vertices } v, \text{ plaquettes } p \}$$

We can repackage $\mathcal{H}$ as the "ground state space" of

$$H = \sum_v (I - X_v) + \sum_p (I - Z_p)$$

H is a nonnegative Hermitian operator

i.e. $\mathcal{H} = \ker \mathbf{H}$.

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\text{spec } X = \{-1, 1\}$

$I - X = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$\text{spec } I - X = \{0, 2\}$

Similarly,

$\text{spec } I - X_v = \{0, 2\}$

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\text{spec } Z = \{-1, 1\}$

$I - Z = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

$\text{spec } I - Z = \{0, 2\}$

$\text{spec } I - Z_p = \{0, 2\}$

X_v's and Z_p's commute, hence so do $I - X_v$'s and $I - Z_p$'s

Physics intuition: $\mathcal{H}$ consists of "vacuum" or "zero-energy" states w.r.t. Hamiltonian H.

Eigenvectors of H corresponding to NON-zero eigenvalues?

spec H: $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$

The eigenspace is the state space for (pairs of) "elementary particles" of this system...

In fact, spec H: $0 < 4 < 8 < \cdots$

What is E_{λ_1}, the eigen-space corresponding to λ_1?

More intuition: suppose $X_v|\psi\rangle \neq |\psi\rangle$.
We might say that $|\psi\rangle$ has a (nonzero) "charge" or "particle" at vertex v.

Similarly, if $Z_p|\psi\rangle \neq |\psi\rangle$, might say $|\psi\rangle$ has a (nonzero) flux through P, or $|\psi\rangle$ has a "vortex" on plaquette P.

A lowest energy state, i.e. $|\psi\rangle \in E_1$ must violate as few of the constraints $X_v|\psi\rangle = |\psi\rangle$, $Z_p|\psi\rangle = |\psi\rangle$ as possible.

B/c $\prod_v X_v = Id = \prod_P Z_p$, $|\psi\rangle$ either violates exactly two X_v's or two Z_p's.

Recall:

If c is a loop in 1-skeleton,

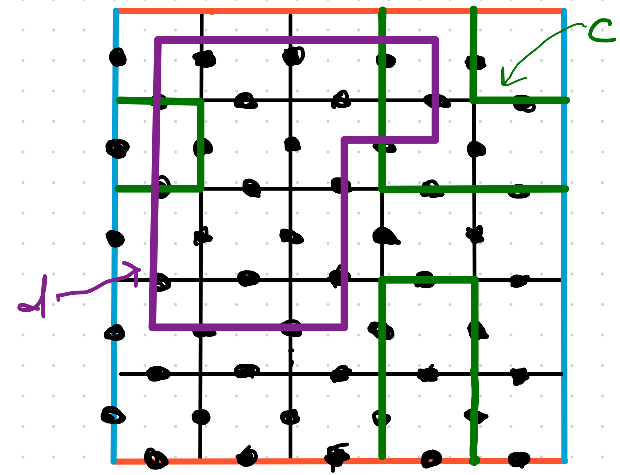
define

$$Z_c = \prod_{e \in c} Z_e$$

If d a loop in dual 1-skeleton,

define

$$X_d = \prod_{e \in d} X_e$$



The loop operators generate possible errors of the code.

A loop operator implements an undetectable and nontrivial error if and only if the loop is nontrivial in

$$H_1(S^1 \times S^1; \mathbb{Z}/2)$$

String operators:

If c is a path in 1-skeleton,

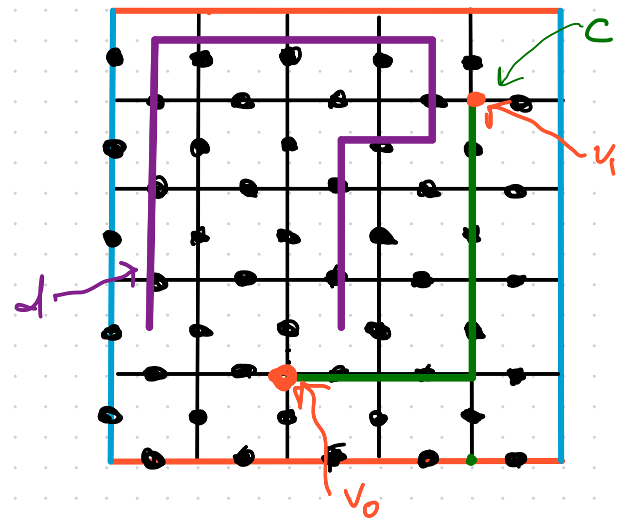
define

$$Z_c = \prod_{e \in c} Z_e$$

If d a path in dual 1-skeleton,

efine

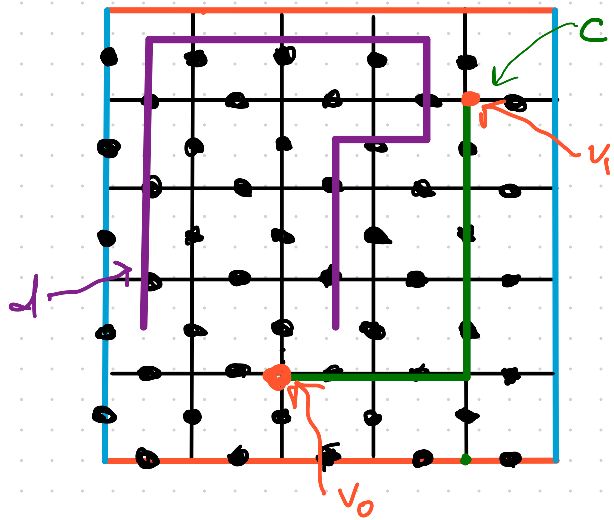
$$X_d = \prod_{e \in d} X_e .$$



$X_{v_0} Z_c |\psi\rangle = -Z_c |\psi\rangle$

If $|\psi\rangle \in \mathcal{H}_1$ then $Z_c|\psi\rangle$ and $X_d|\psi\rangle$ are in

E_{λ_1}. For instance, $Z_c|\psi\rangle$ will violate the two X_v's at ends of c.



On the other hand

$$X_{v_0} Z_c |\psi\rangle = -Z_c|\psi\rangle$$

$$X_{v_1} Z_c |\psi\rangle = -Z_c|\psi\rangle$$

$$X_v Z_c|\psi\rangle = Z_c|\psi\rangle$$

For any $v \neq v_0, v_1$

and

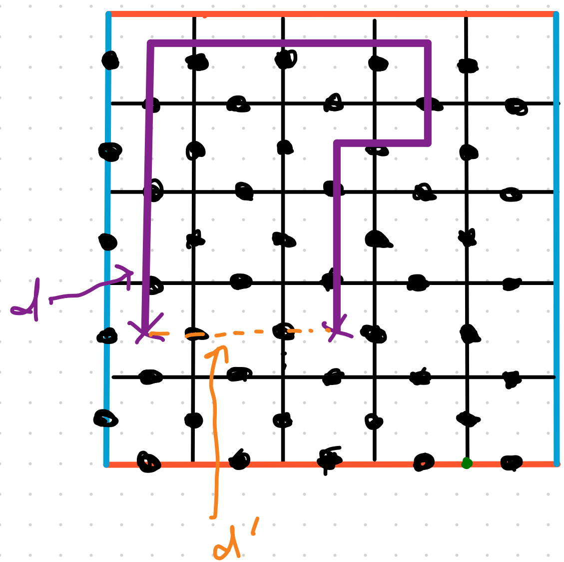
$$Z_p Z_c |\psi\rangle = Z_c|\psi\rangle$$

For all P.

Note: IF $|\psi\rangle \in \mathcal{H}$,

$$Z_c|\psi\rangle = Z_{c'}|\psi\rangle \text{ iF and only iF } c + c' = 0 \text{ in } H_1(S^1 \times S^1; \mathbb{Z}/2)$$

Similarly for X_d's.



More generally, we need $c \sim c'$ rel endpoints

i.e, for other "surface codes"

Kitaev's ($100,000,000?) idea:

Introduce a small number of particles onto the surface, and move them around in controlled ways in order to intentionally manipulate a codestate of the toric code. Because nontrivial operations occur only after doing something "topologically nontrivial," the probability of implementing the wrong operation can be made small without much overhead.

In other words: "Fault tolerance from hardware" if you can implement the toric code Hamiltonian in a lab.

Two related ideas for how to process information topologically using toric code:

1. braiding

2. "Dehn twisting"

Braiding in Kitaev's model

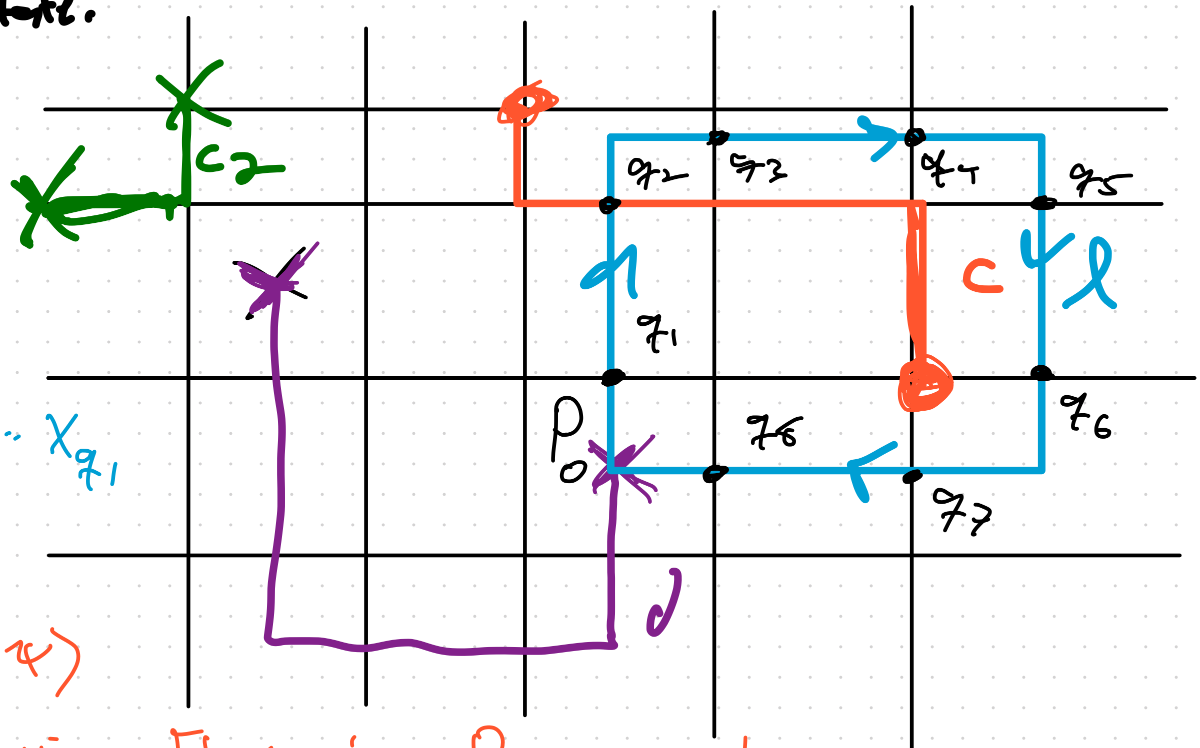
$|\psi\rangle \in \mathcal{H}$

Start w/ state:

$X_d Z_c |\psi\rangle$

$\Uparrow$

E_{x_2}



$X_\ell = X_{q_8} X_{q_7} \cdots X_{q_1}$

Applying X_ℓ to $X_d Z_c |\psi\rangle$ is like moving flux in P_0 around one of the charges at the end of the C string

$$X_\ell (X_d Z_c |\psi\rangle) = (-1)\, X_d Z_c |\psi\rangle$$

Note: this is true only because of the charge at ~~there~~ end of the string c.

Weird! Moving flux along a loop around a charge implements a nontrivial change to the state.

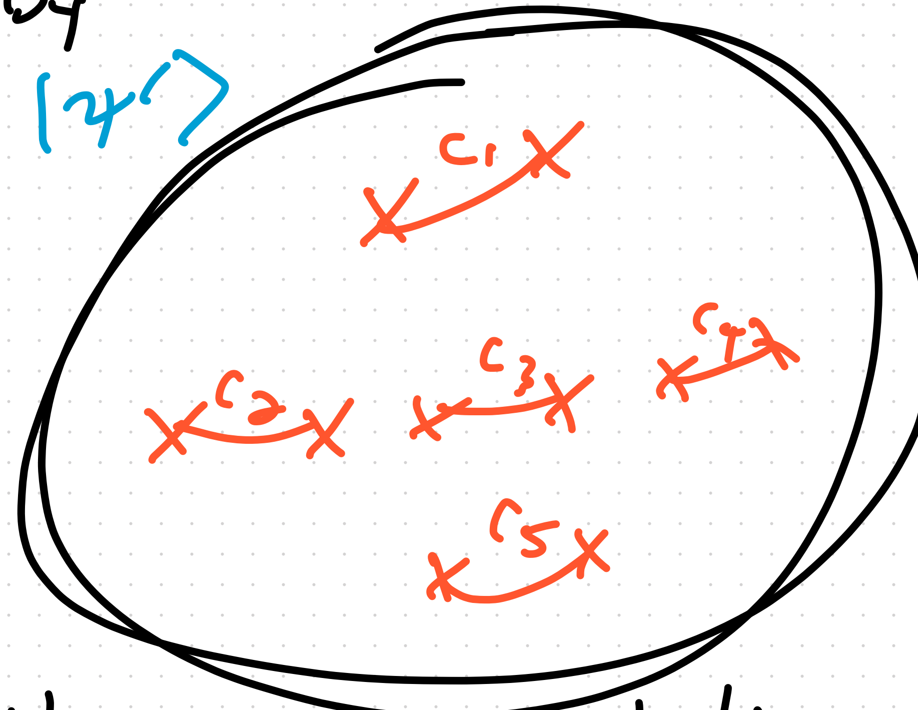
The elementary violations of the stabilizer (i.e. the particles) have "non trivial" braiding statistics.

Such particles are called <u>anyons</u>.

Start w/ $|\psi\rangle \in \mathcal{H}$.
"Create particles" by

$$Z_{c_1} Z_{c_2} \cdots Z_{c_5} |\psi\rangle = |\psi'\rangle$$

This introduces 10 particles and puts us inside E_{λ_5}



Loop operators yield a representation of 10 strand braid group!

$|\gamma\rangle \in E_{\lambda_5}$

$V = $ Loops: $|\gamma\rangle \subseteq E_{\lambda_5}$

Unitary Representation

$$B_{10} \longrightarrow U(V).$$

"Dehn twisting"

$|00\rangle \mapsto Z_{c_0}|00\rangle \mapsto Z_{c_1}Z_{c_0}|00\rangle$

$= X|0\rangle \otimes |0\rangle$

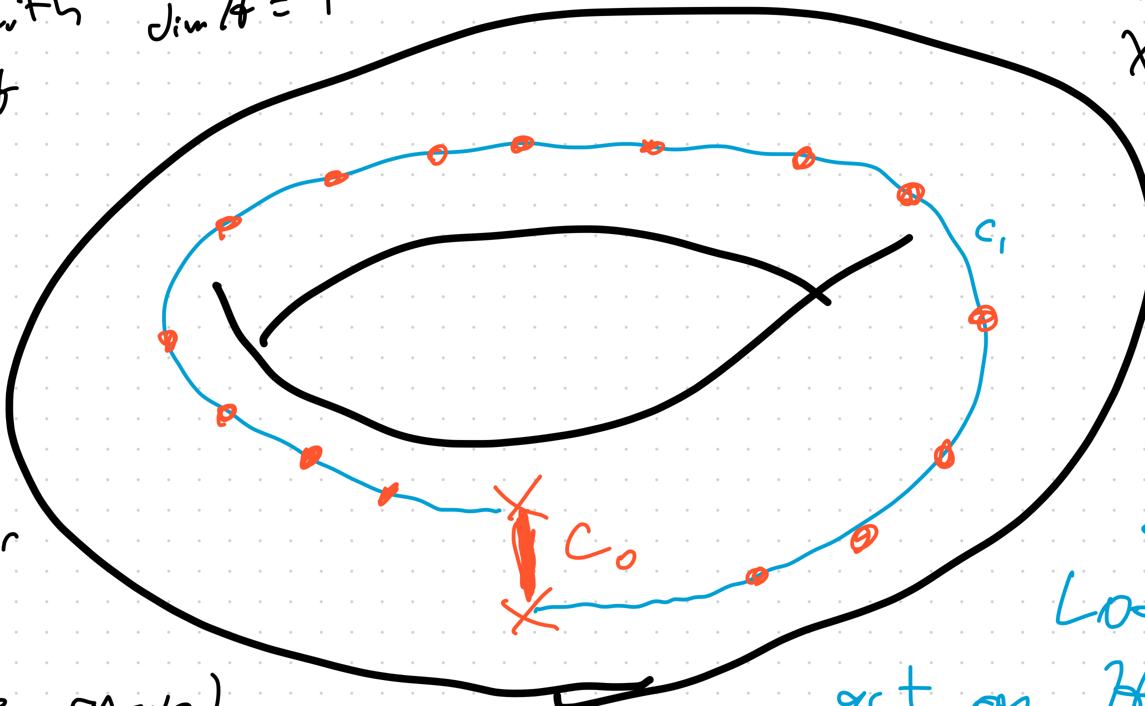
or

$|0\rangle \otimes Z|0\rangle$

Start with $\dim \mathcal{H} = 4$

$|\psi\rangle \in \mathcal{H}$

Manipulate $|\psi\rangle$ into another state in $\mathcal{H}$ by creating pair of anyons, move one around torus, then annihilate



Recall:

Loop operators act on $\mathcal{H}$ like the Pauli X's and Z's

$\mathcal{H} \cong \mathbb{C}^2 \otimes \mathbb{C}^2 = \text{span}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

loop operators act on $\mathcal{H}$ by

$$X \otimes \text{Id}, \quad Z \otimes \text{Id}$$

$$\text{Id} \otimes X, \text{ or } \text{Id} \otimes Z$$

So, can process quantum info in $\mathcal{H}$ by applying loop operators.

Problem with toric code?

With toric code, we can only implement X's and Z's on codespace using loop operators. So "Dehn twist" idea is insufficient to generate quantum universal op's on $\mathcal{H}$

Similar issues for braiding

Are there other, similar setups, but where the topologically protected operations are powerful enough to implement a quantum universal gate set?

Yes!

This is what Freedman - Larsen - Wang prove.

II. TQC and TQFT go hand in hand.

Once-extended (2+1)-dimensional TQFTs provide a language to abstract away the combinatorial aspects of Kitaev's proposal, and focus on the topology of anyons and their interactions