Meeting 11.1: Topological quantum computing, II

I. Quantum circuits inside extended TQFTs

II. Which TQFTs are BQP-universal?

I. Quantum circuits inside extended TQFTs

Last time: string and loop operators in toric code are insufficient to build a universal quantum computer.

However, the idea is useful because string and loop operators are "topologically protected" operations.

Are there variations of toric code construction whose topologically protected operations are powerful enough to approximate arbitrary quantum circuits?

Freedman, Larsen, Wang (2002) showed answer is

YES.

Specifically, they use the "Jones TQFT with $q = e^{2\pi i/5}$".

This TQFT has other names:

- $SU(2)$ Chern-Simons at level 3
- Witten-Reshetikhin-Turaev theory for $U_q sl_2$, $q = e^{2\pi i/5}$
- expected anyon statistics for fractional quantum Hall effect at certain filling fraction...

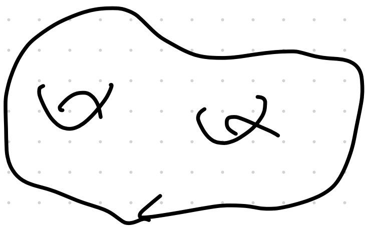
We're going to work through the Freedman-Larsen-Wang construction this week.

First, we need to understand formal properties of unitary once extended (2+1)-dimensional TQFTs.(*) This will allow us to formulate general conditions that allow a TQFT to be used to simulate quantum circuits.

Then, we will need to check the Jones TQFT satisfies these conditions.

(*) I'll say some things later about these restrictions. For now, TQFTs are all unitary.

RECALL: a (non-extended/Atiyah style) unitary (2+1)-dimensional TQFT is a $\otimes$-functor from Cob(2+1) to Hilb ← $\otimes$-category of finite dimensional Hilbert spaces

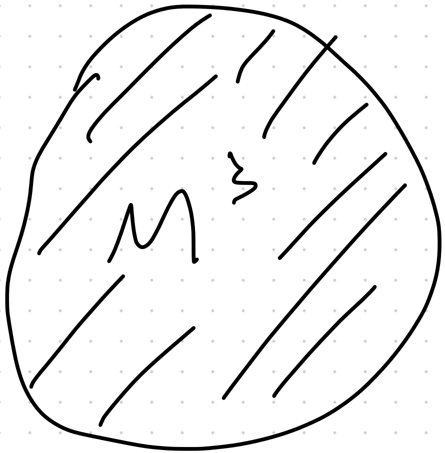


surface $S \mapsto Z(S)$ ← Hilbert space

$M \mapsto Z(M) : Z(S_0) \otimes Z(S_1) \to Z(S_2)$

Reasons to like Atiyah style (2+1)-dim TQFTs:

• Good source of $\mathbb{C}$-valued invariants of closed 3-manifolds



$$\begin{tikzcd} M \arrow[r, mapsto] & Z(M)\colon & Z(\emptyset) \arrow[r] \arrow[d, equal] & Z(\emptyset) \arrow[d, equal] \\ & & \mathbb{C} & \mathbb{C} \end{tikzcd}$$

$Z(M)$ is a liner map $\mathbb{C} \to \mathbb{C}$,
hence $Z(M) \in \mathbb{C}$.

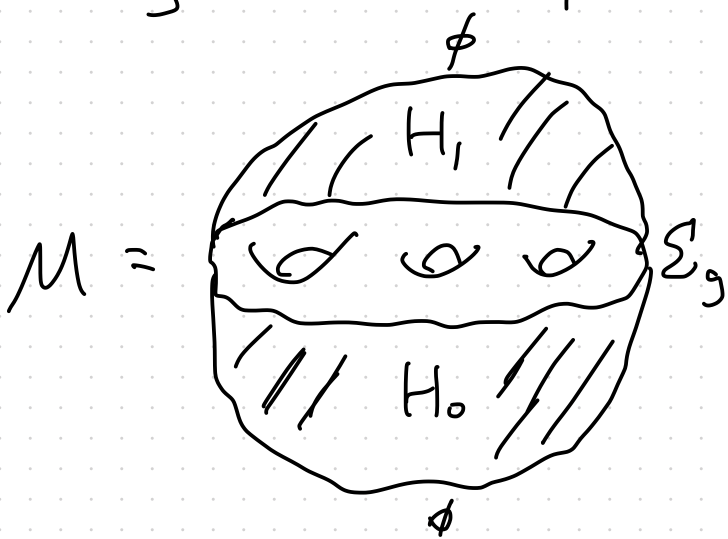
If M^3 is closed,
then $\partial M = \emptyset$

If $Z(M^3) \neq Z(N^3)$, then
$M^3 \not\cong N^3$.

Reasons to like Atiyah style (2+1)-dim TQFTs:

- TQFT axioms allow us to compute these invariants via "cut and paste", e.g. from a Heegaard splitting.

If $M = H_0 \underset{\partial H_0 \cong \partial H_1}{\sqcup} H_1$ where H_0 and H_1 are two genus g handlebodies, we can "divide and conquer":



$$Z(M) = Z(H_1) \circ Z(H_0)$$

Get representations of mapping class groups of closed surfaces from a (2+1)-TQFT. Called quantum representations of mapping class groups.

If S is an oriented surface,

$$MCG(S) := \mathrm{Homeo}_+(S) / \text{isotopy}$$

Intuition: $MCG(S)$ is "orientation-preserving homeomorphisms modulo isotopy."

$$\begin{aligned} MCG(S) &\cong \mathrm{Homeo}_+(S) / \text{homotopy} \\ &\cong \mathrm{Diffeo}_+(S) / \text{isotopy} \\ &\cong \mathrm{Homeo}_+(S) / \text{Normal subgroup of homeomorphisms isotopic to identity} \end{aligned}$$

$f, g : S \to S$ are isotopic if there exists

$$H : S \times [0,1] \to S$$

such that:

(i) $H(x, 0) = f(x) \quad \forall x \in S$

(ii) $H(x, 1) = g(x) \quad \forall x \in S$

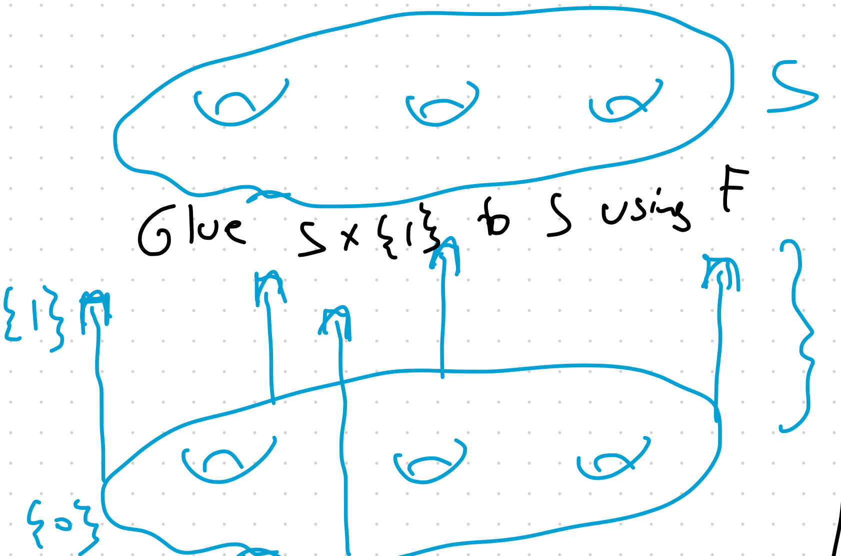
(iii) $H(x, t)$ is a homeomorphism $S \to S$ for each fixed t.

(iv) H is continuous.

Building quantum reps from TQFT?

Given Z, surface S, and homeomorphism $F: S \to S$, can build $Z(F): Z(S) \to Z(S)$

by taking the mapping cylinder of F:



$$Z(F) := Z(M_F)$$

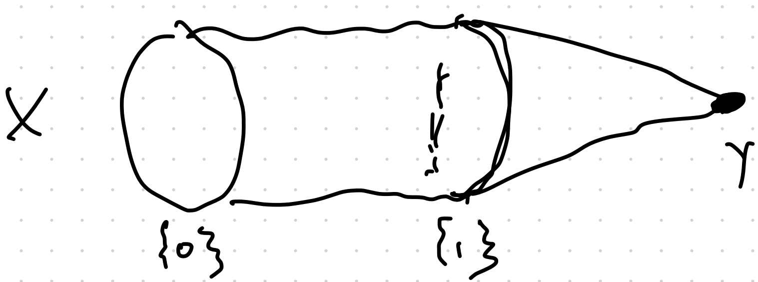
If f, g are isotopic, $M_f \cong M_g$, thus, by axioms of TQFT,

$$Z(f) = Z(M_f) = Z(M_g) = Z(g)$$

Mapping cylinder of $f: X \to Y$

$$M_f := X \times [0,1] \sqcup Y \Big/ (x,1) \sim f(x)$$

e.g. $X = S^1$, $Y = \{pt\}$, f constant map:



Given TQFT $Z: \mathrm{Cob}(2+1) \to \mathrm{Vec}$,

For each surface S, we get a representation

$$Z: MCG(S) \to GL(Z(S)).$$

Called a quantum representation.

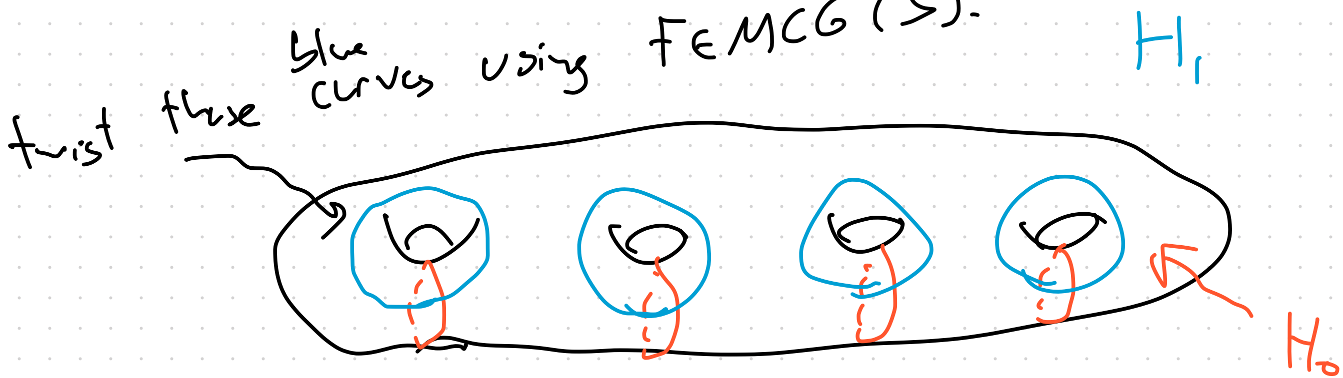
Warning: w/ more careful axioms, might only get a projective representation...

If TQFT is unitary, then the quantum representations are unitary. (Later, they may only be projective unitary, but that will suffice.)

$$Z : MCG(S) \to PU(Z(S))$$

Can calculate 3-manifold invariants using the quantum representations

Suppose M is a 3-fold formed by "twisting" the standard Heegaard splitting of S^3 by a genus g surface



Lemma: if $c_1, \dots, c_g$ are a complete disk system, and $[F] \in MCG(S)$ is represented by F, then $f(c_1), \dots, f(c_g)$ forms another complete disks system, independent of representative.

Then (up to a sign error in exponent)

$$Z(M) = Z(H_1) \circ Z(F)^{-1} \circ Z(H_0).$$

This expresses $Z(M)$ in terms of the quantum representation.

Rough pass at TQC:

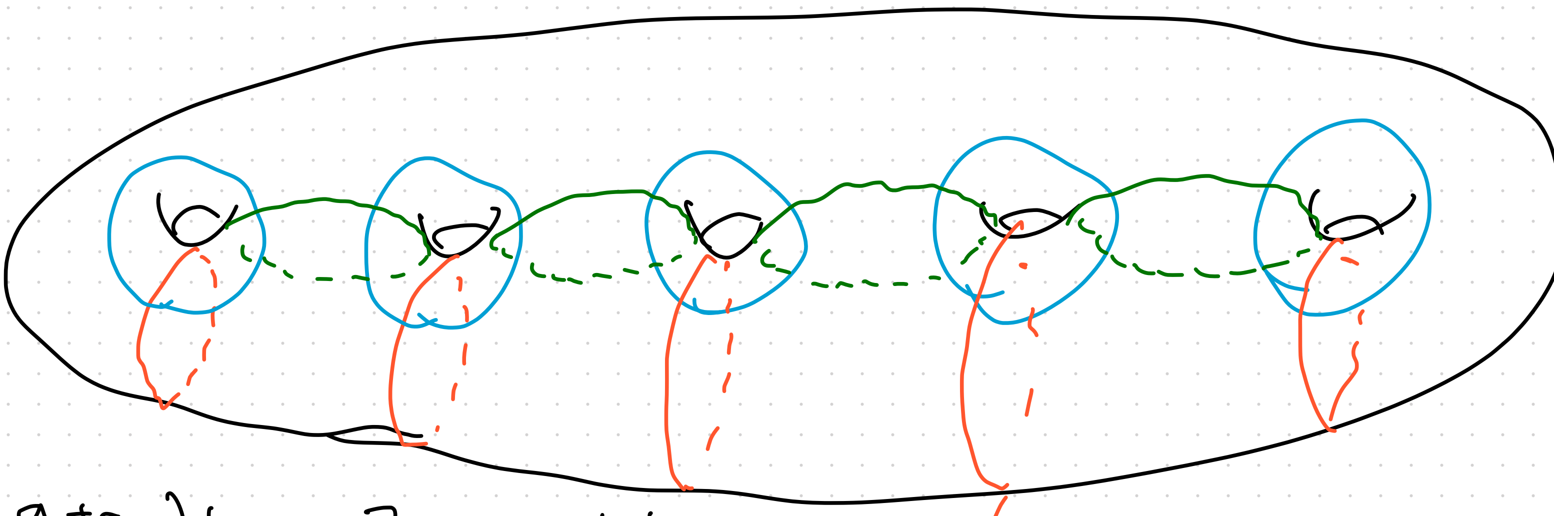
Find a unitary TQFT whose quantum representation for some surface S is dense inside $PU(Z(S))$.

Interpretation: • Hilbert space $Z(S)$ is quantum memory

• Action of a mapping class $f \in MCG(S)$ yields a "quantum circuit" $Z(f): Z(S) \to Z(S)$.

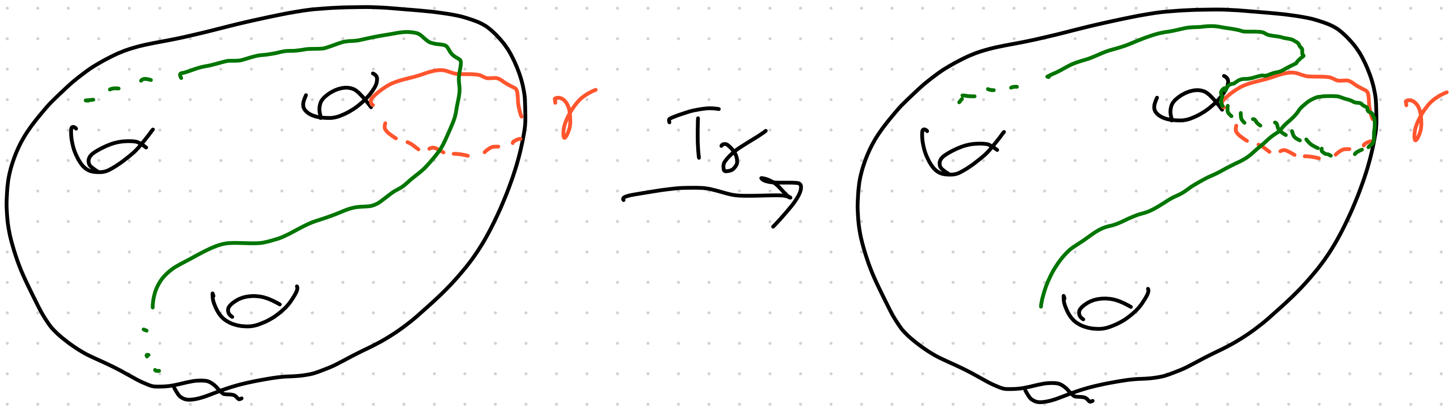
to make precise requires choosing generators of $MCG(S)$.

Theorem (Lickorish–Wallace) $MCG(S)$ is generated by Dehn twists along a specific set of finitely many simple closed curves:



$g + g - 2 + g = 3g - 2$ total generators

Dehn twist

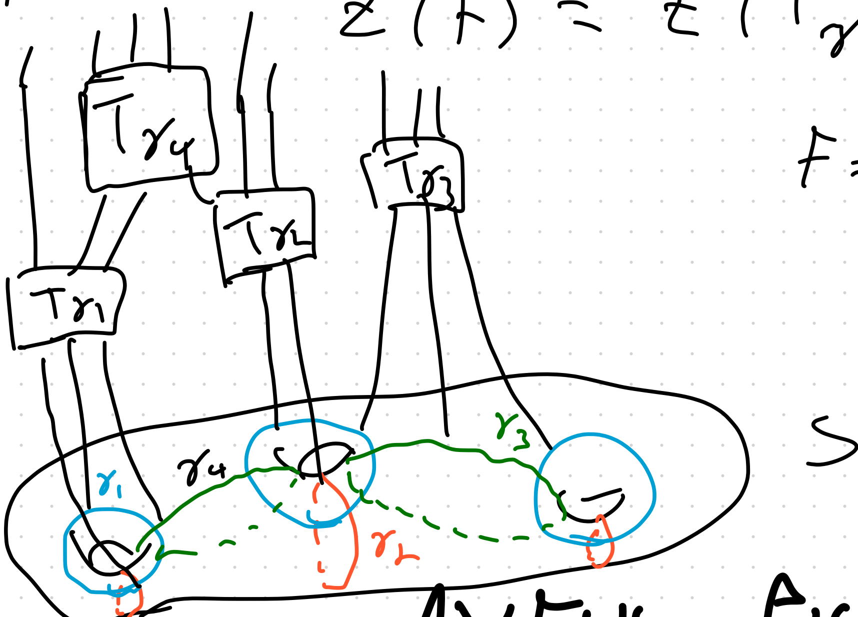


T_γ: cut $N(\gamma)$, and glue back in with a twist

Suppose $Z : MCG(S) \to PU(Z(S))$ is dense, and let $F = \prod_{j=1}^{\ell} T_{\gamma_j}$. Applying Z to F yields

$$Z(F) = Z(T_{\gamma_1}) \circ Z(T_{\gamma_2}) \circ \cdots \circ Z(T_{\gamma_\ell})$$

$$F = T_{\gamma_4} \circ T_{\gamma_3} \circ T_{\gamma_2} \circ T_{\gamma_1}$$



AWFUL PICTURE

Using quantum representation of $MCG(S)$, we could try to simulate circuits.

However, can't be made correct yet, b/c need to decompose $Z(S)$ into tensor products of subspaces in order to "localize" qubits into different regions of surface.

Need extended TQFT!