Meeting 11.2: Topological quantum computing, III

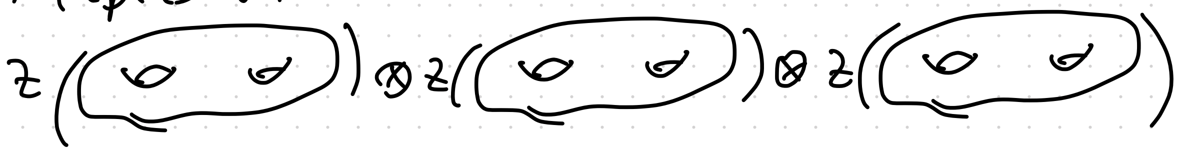
I. Quantum circuits inside extended TQFTs, continued

II. Which TQFTs are BQP-universal?

Last time: can try to use quantum representation of mapping class group of (closed) surface determined by a unitary TQFT to process quantum information.

Problem: don't have any clear way to decompose Hilbert space $Z(S)$ into tensor product of subspaces.
So, unclear how to encode quantum circuits...

One (unhelpful) idea: use disconnected surfaces?
Quantum representation won't generate entanglement....



Solution: use extended TQFT!

EXTENDED TQFTs have even nicer cut and paste properties

In addition to computing $Z(M^3)$ by cutting M^3 along a surface, we can cut surfaces along curves to compute their state spaces.

To make precise, we will assume extended TQFTs come equipped with a finite set of "colors" $C = \{1, 2, \dots, r\}$.

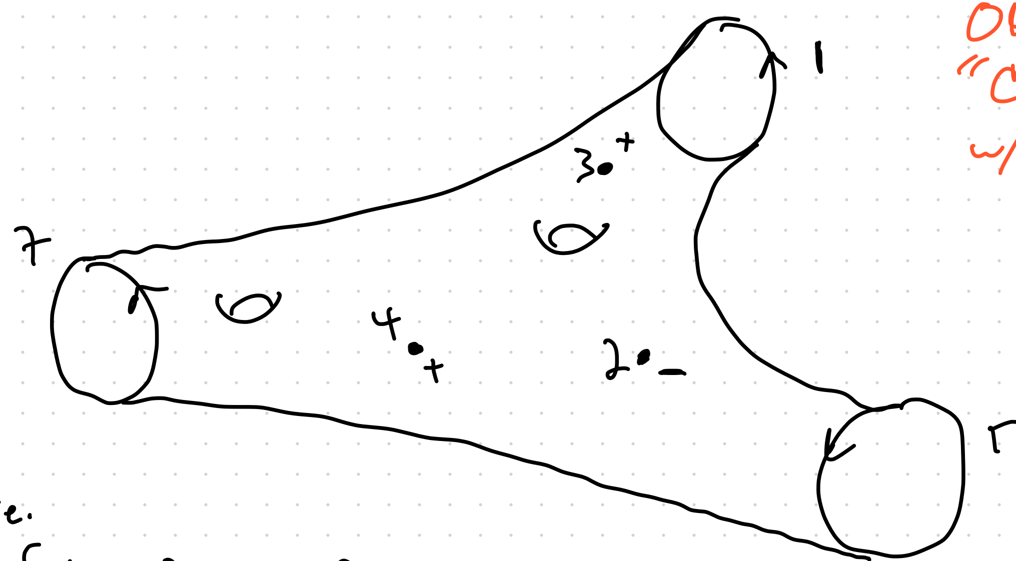
We then define the C-colored, once extended (2+1)-dimensional cobordism category $C\text{-}Cob(2+1)$.

Contains $Cob(2+1)$ as a subcategory.

Note: what follows is imprecise and incorrect, probably. Why? Don't want to define extended TQFT or modular tensor category in full detail...

C-Cob$(2+1)$ includes new objects:

surfaces with (oriented ....) boundary and oriented marked points, with all boundary components and marked points



Objects are "C-colored surfaces w/ marked points"

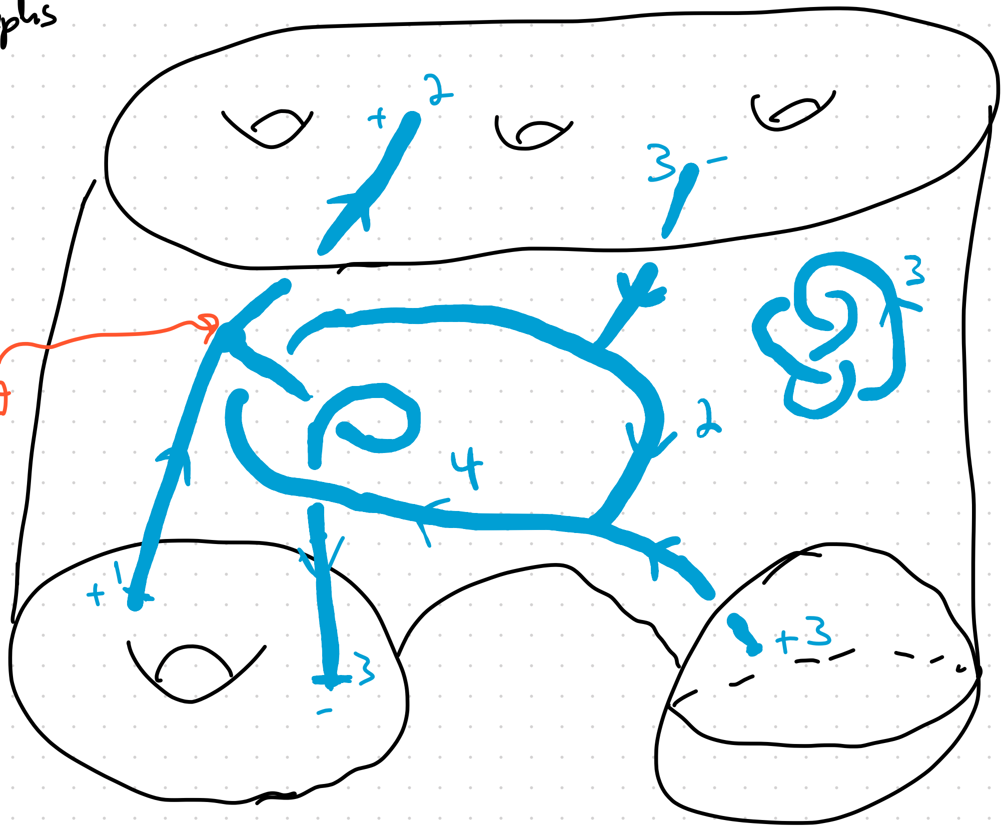
$r = 10$, i.e.

$C = \{1, 2, 3, \dots, 10\}$

And new morphisms:

3-manifolds with properly embedded, $\mathcal{C}$-colored trivalent, oriented ribbon graphs

(Need to allow 3-manifolds with "corners.")

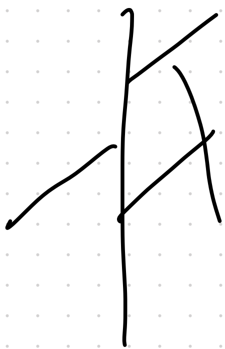


Technically also need to color trivalent vertices by another color set...

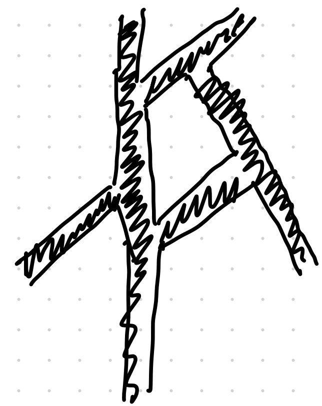
Note:
Morphisms only compose when boundary colorings are compatible

Ribbon graph?

Normal graph

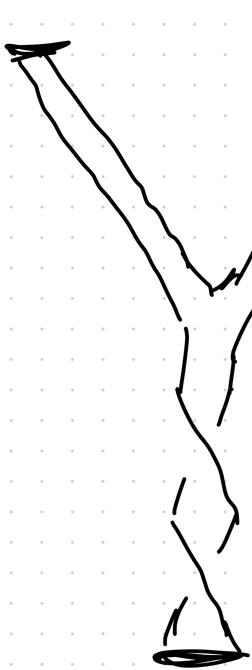


Ribbon graph



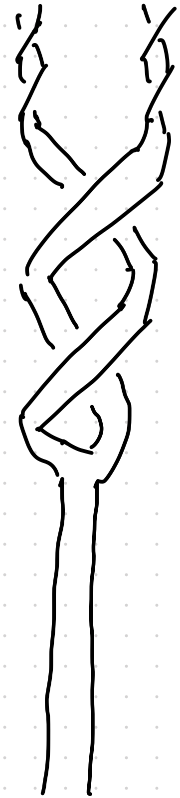
Why? Up to isotopy rel boundary

 $\neq$ 



$\cong$

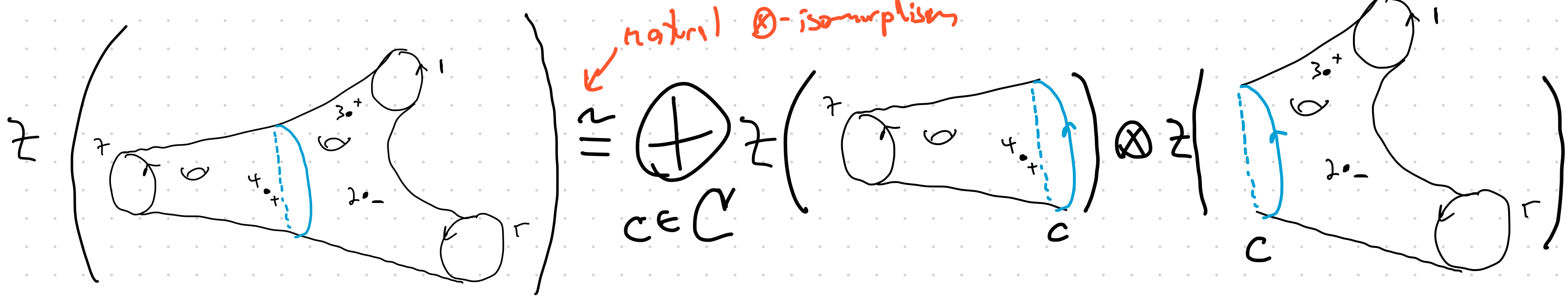
"ribbon twist relation"

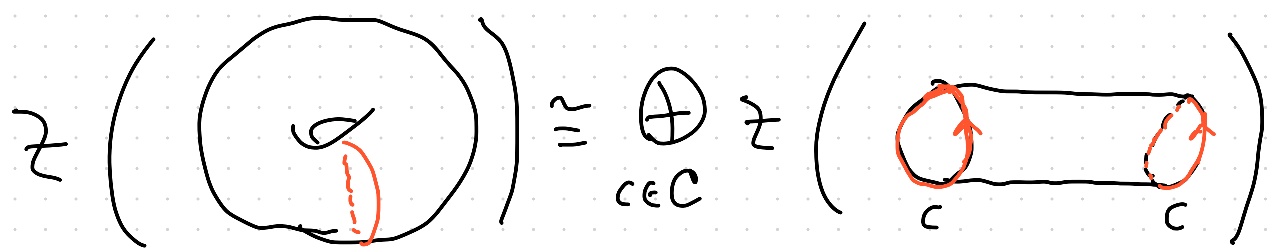


As for usual/unextended TQFTs, an extended unitary TQFT is a $\otimes$-functor

$$Z : \mathcal{C}\text{-Cob}(2+1) \longrightarrow \mathrm{Hilb}.$$

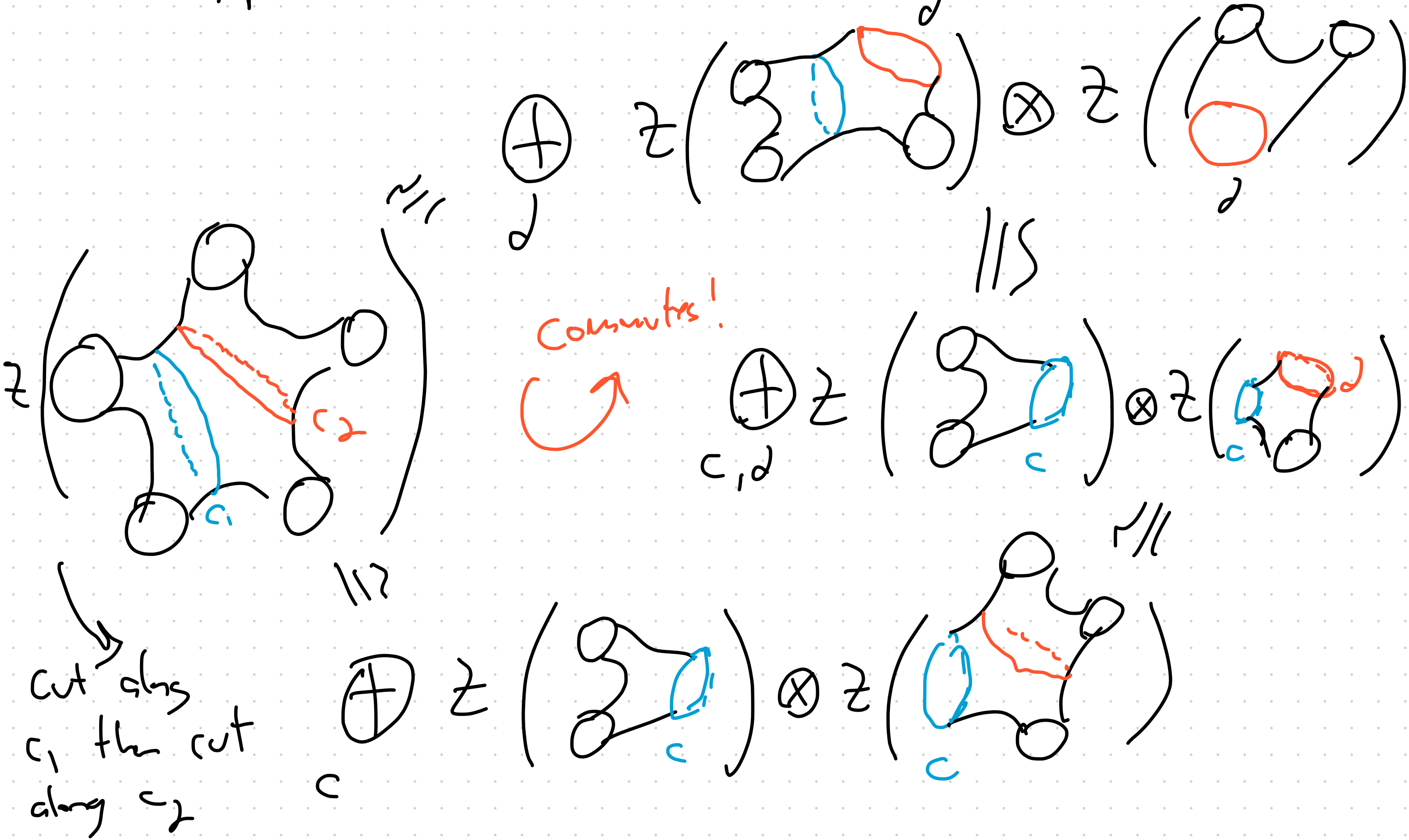
In addition to all the axioms for Atiyah TQFT, an extended TQFT includes axioms that require functoriality w.r.t. cutting/pasting of $\mathcal{C}$-colored surfaces. Most important for our purposes:



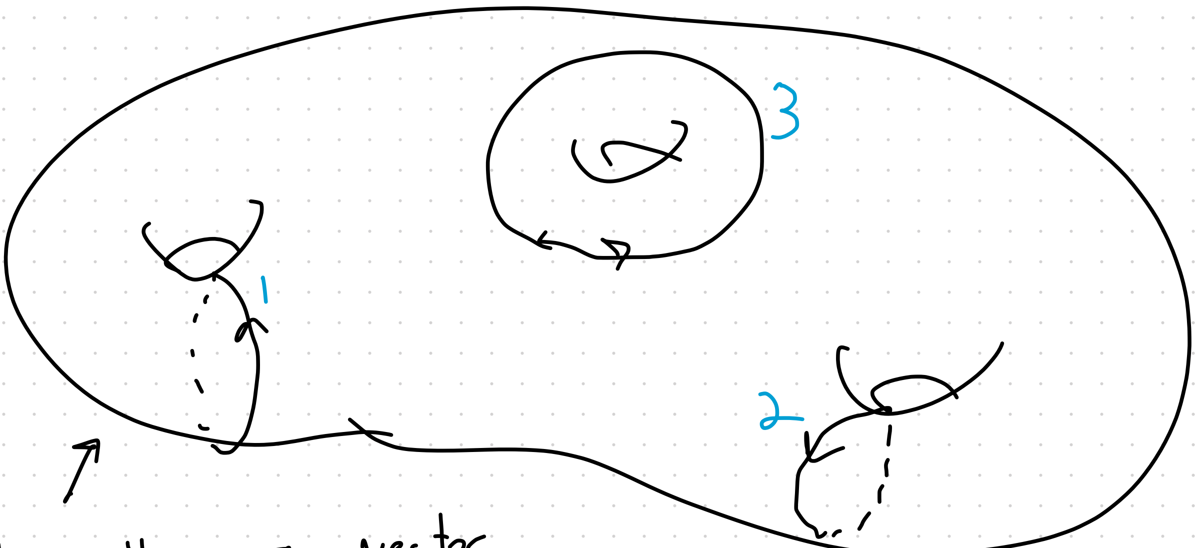


GLUING AXIOM

Cutting/pasting "commute":



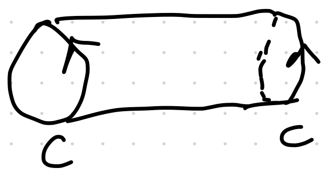
Gluing axiom allows to specify elements of $Z(S)$ by labelling a complete disk system of S with orientations and elements of $\mathcal{C}$



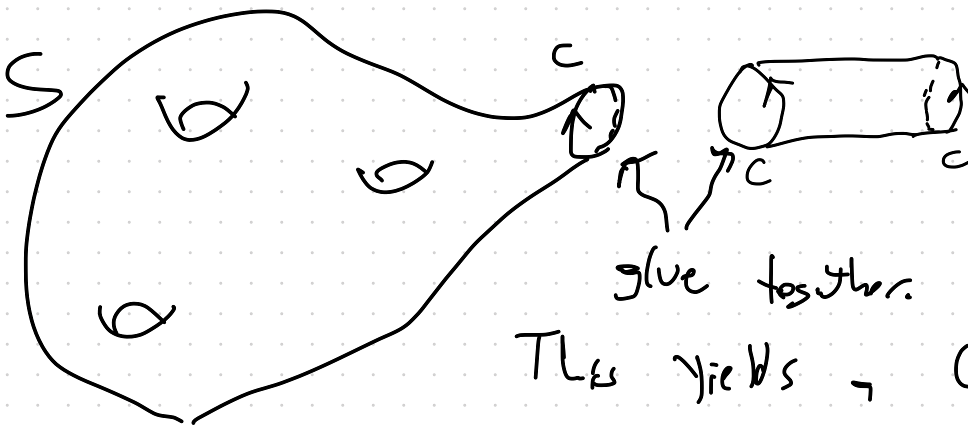
This yields a vector

$Z\Big($  $\Big)$

Note, e.g.

$Z\Big($  $\Big)$ is 1-dimensional.

Why?



This yields a C-colored surface homeomorphic to S.

Thus,

$Z(S) \cong Z(S) \otimes Z($  $)$

must be 1-dimensional!

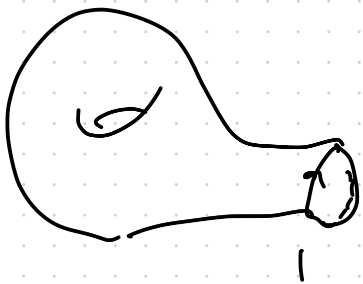
Another axiom: the color 1 is special

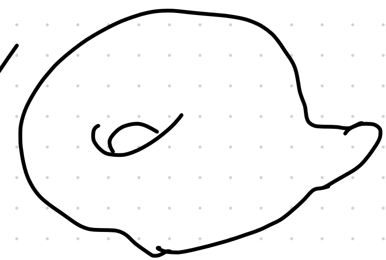
$Z\Bigg($ 

$\Bigg) \cong Z\Bigg($ 

$\Bigg)$

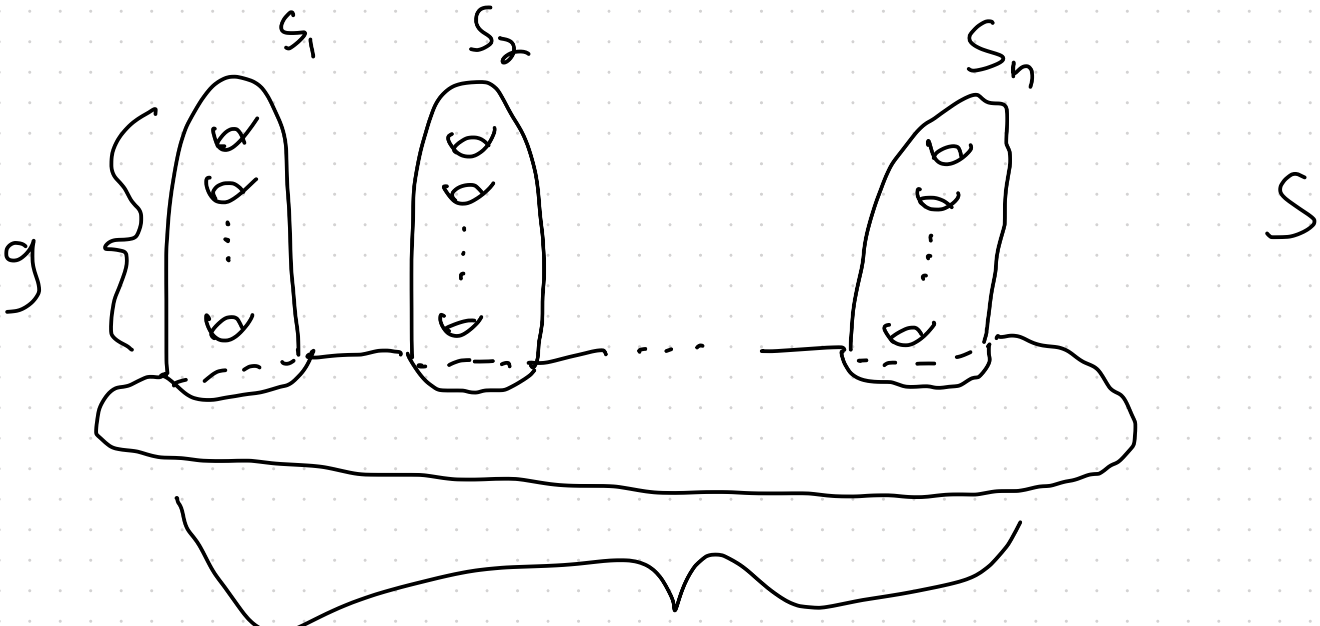
remove points labelled by 1

$Z\Bigg($ 

$\Bigg) \cong Z\Bigg($  $\Bigg)$

cap off boundaries labelled by 1

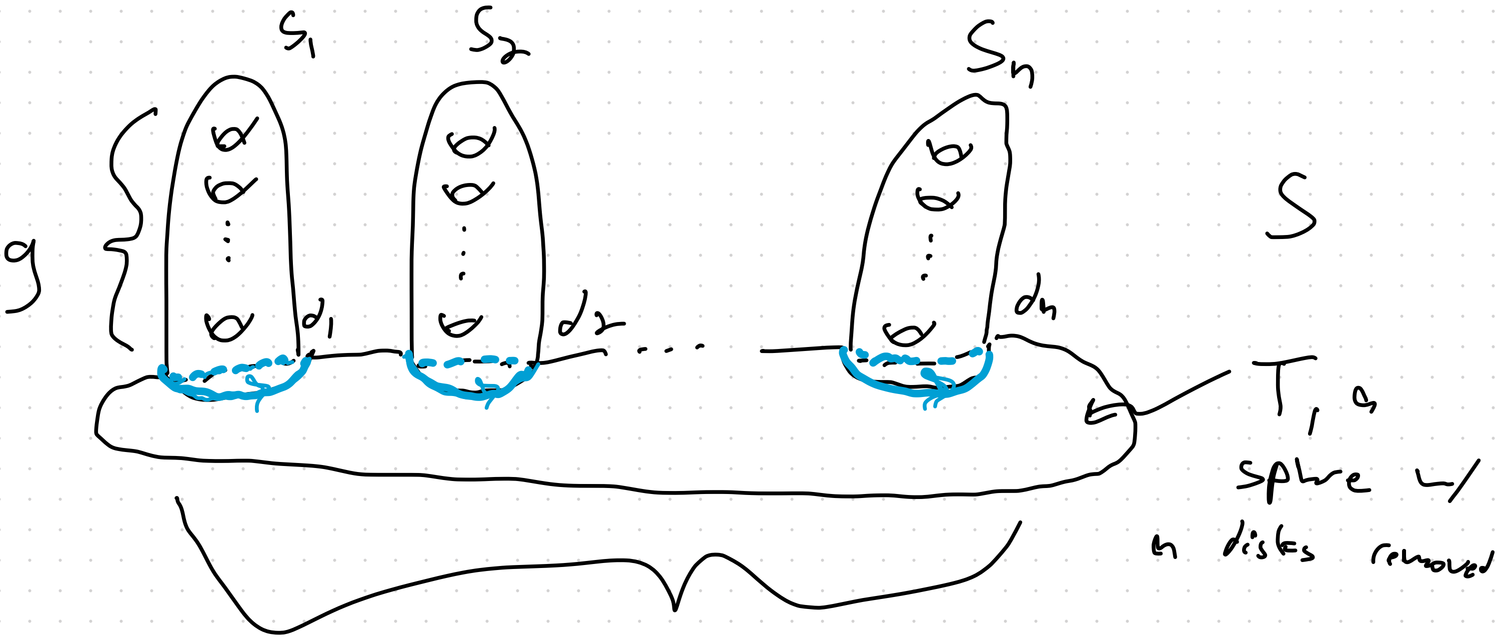
Gluing axioms allow us to build models of circuits. Here's one based on closed surfaces.



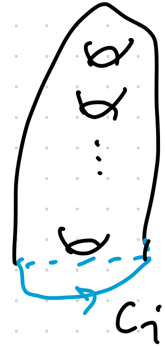
Given Z and S, we "localize" the quantum rep of $MCG(S)$ along the S_i's.

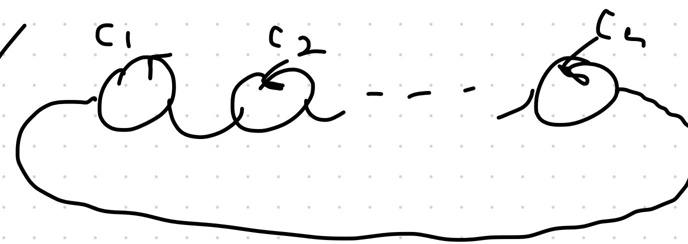
Via gluing, state in $Z(S)$ lies in a subspace of the form

$$Z(S_1, c_1) \otimes Z(S_2, c_2) \otimes \cdots \otimes Z(S_n, c_n) \otimes Z(T, c_1, c_2, \ldots, c_n)$$



where

$Z(S_i, c_i) = Z\Big($  $\Big)$

$Z(T, c_1, \dots, c_n) = Z\Big($  $\Big)$

$Z(S)$ is spanned by subspaces of form

$$Z(S_1, c_1) \otimes Z(S_2, c_2) \otimes \cdots \otimes Z(S_n, c_n) \otimes Z(T, c_1, c_2, \ldots, c_n)$$

as we vary c_i's in $C = \{1, 2, \ldots, r\}$,

Setting up circuits

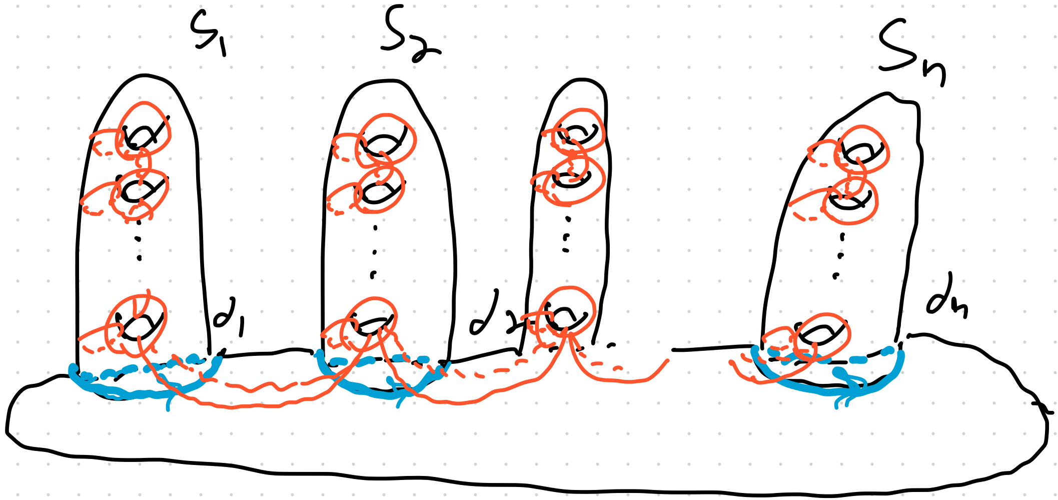
Various ways to do it...

Let's fix one subspace, i.e. this one:

$$Z(S_1, c) \otimes Z(S_2, c) \otimes \cdots \otimes Z(S_n, c) \otimes Z(T, c, c, \ldots, c)$$

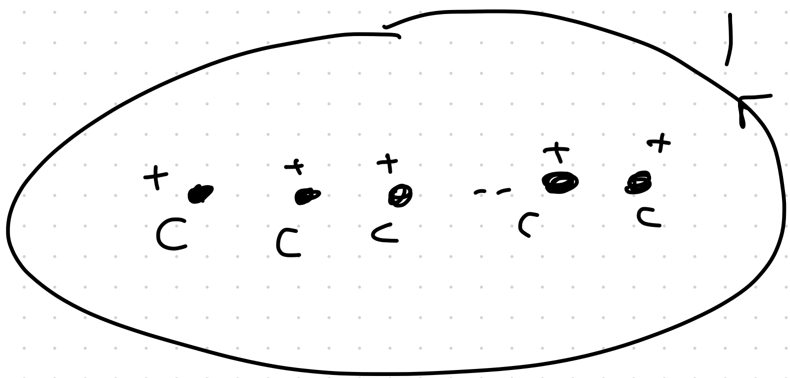
Look at $\Gamma = \Gamma(c) \leq MCG(S)$, the subgroup that preserves this subspace.

Dehn twists along orange curves generate MCG(S)



Another approach: use braids

Consider disk $D_n(c)$ n points all colored by c and boundary colored by 1:



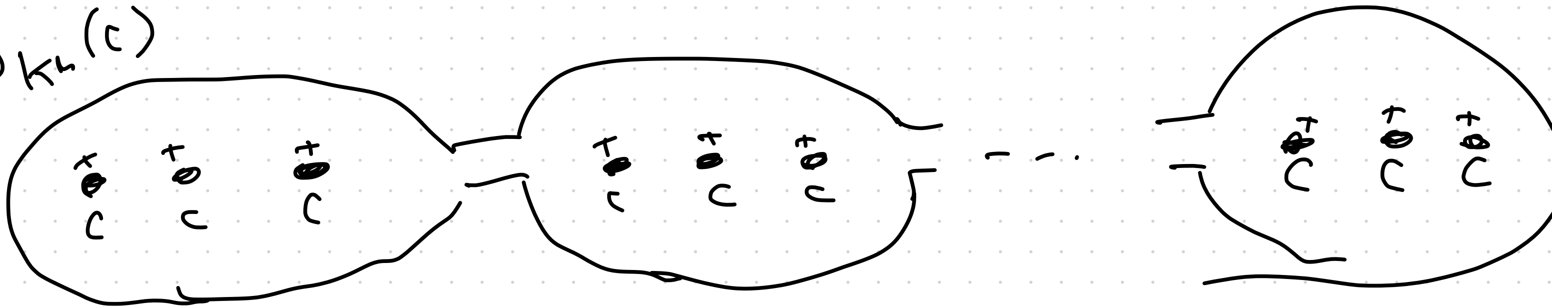
$D_n(c)$

B_n acts on $Z(D_n(c))$

Fix k, and consider n copies of $D_k(c)$ glued together along boundary

e.g. $k=3$

$D_{kn}(c)$



$B_{kn} \curvearrowright D_{kn}(c)$.

$Z(D_{kn}(c))$ contains of $\otimes$'s of $Z(D_{kn}(c))$ ...

$$Z(D_k(c)) \otimes \cdots \otimes Z(D_k(c)) \subseteq Z(D_{kn}(c)).$$

Can use subgroup of B_{kn} that preserves this subspace to build circuits.

What circuits can we simulate?

II. Which TQFTs are BQP-universal?

(Should assume TQFT is unitary and extended.)

Interesting question in all dimensions.

Has received most attention in dimension 2+1.

Why?

- In dimensions < 3, too weak....
- In dimensions > 3, poorly understood. Expected to be too weak if "fully extended"...
- In dimension 3, extended TQFTs all come from modular tensor categories via Reshetikhin-Turaev construction. "Combinatorial"-ish

Key question: given Σ, are there colored surfaces S whose quantum representations are dense in $PU(\mathcal{Z}(S))$?

If answer is yes, you can simulate/approximate (via Solovay-Kitaev and additional tricks) arbitrary quantum circuits.