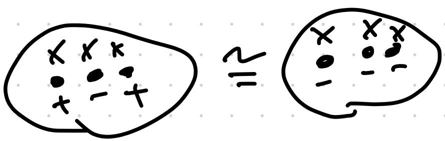
Meeting 12.2: Topological quantum computing, IV

I. Universality from density of quantum representations of braid groups

Fix a $\mathcal{C}$-colored extended unitary TQFT Z and a color $X \in \mathcal{C}$ (secretly, X is a simple object in the unitary modular tensor category determined by Z).

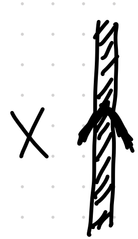
For convenience:

Assume X is "self dual," meaning Z treats colored points

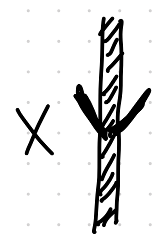


$\overset{+}{\bullet}_X$ and $\overset{-}{\bullet}_X$ as identical

and also



and

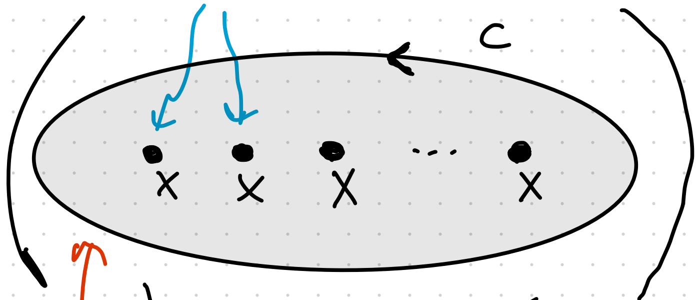


as identical.

E.g. the "special" color 1 is self dual. But we will want $X \neq 1$

Define Hilbert space: $c \in \mathcal{C}$

do not need to orient b/c self dual

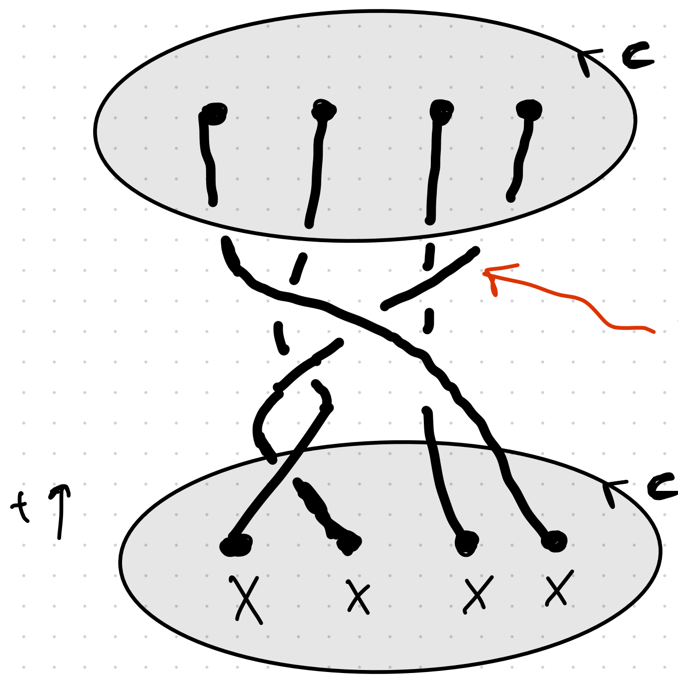
$Z(n; X, c) := Z\Big($  $\Big)$

n points on a disk

this disk w/ n colored points will be denoted D_n^c.

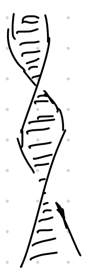
B_n acts on $Z(n; X, c)$:

e.g. $n = 4$



these are ribbons w/ "blackboard framing"

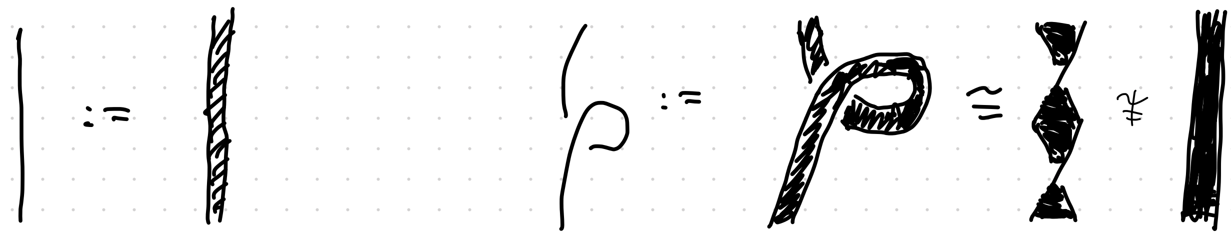
i.e. "no twisting"



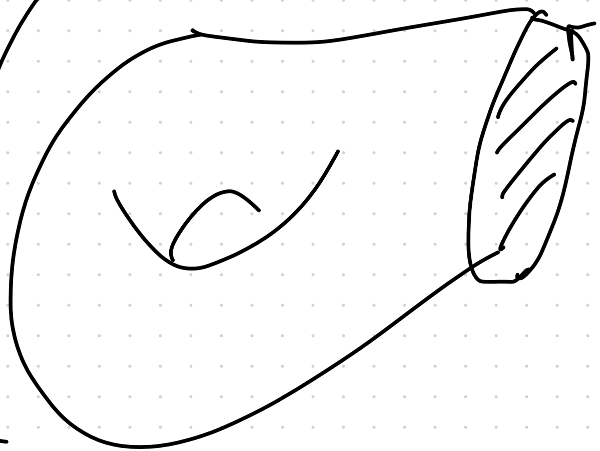
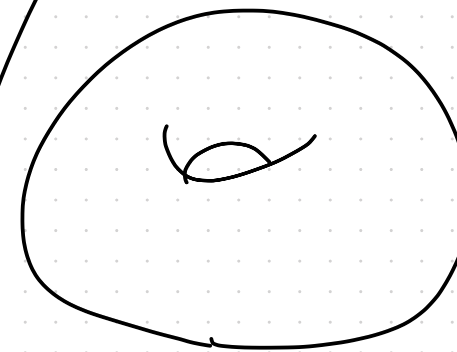
Tangent: blackboard framing convention

Problem: Drawing ribbons is annoying

Solution:

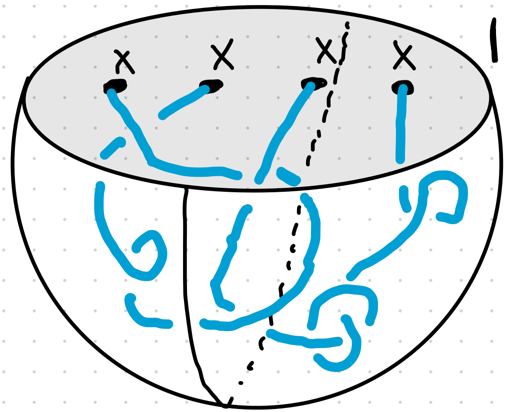


Warning: No Reidemeister I-move!

$Z\Big($  $\Big)^{1} \simeq Z\Big($  $\Big)$

Consider $D'_{2k} \subseteq S_{2k}$, 2-sphere w/ $2k$ colored points, and this sphere as ∂ of a 3-ball B^3

E.g. $k = 2$



Remark: This trick is why we assume X is self-dual.

Every k-tangle T in B^3 yields a linear map

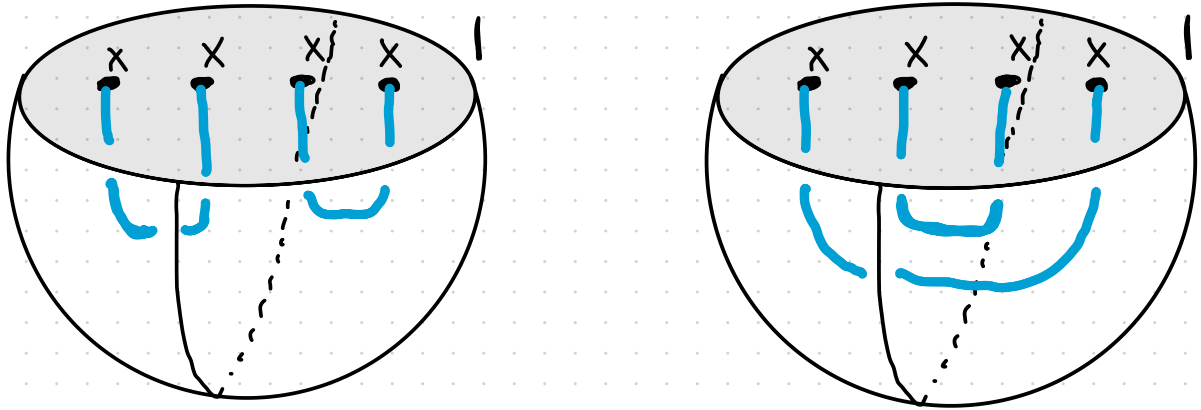
$$Z(T) : \mathbb{C} \longrightarrow Z(S_{2k}),$$

hence, a vector $|T\rangle$ (namely, $Z(T)(1)$) in

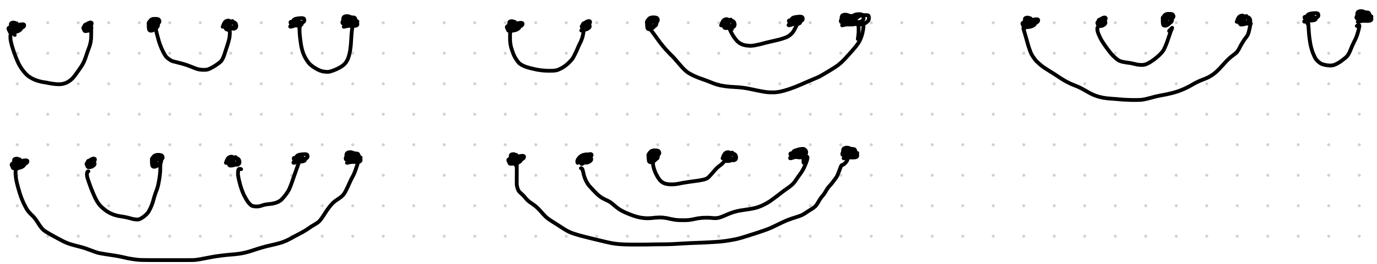
$$Z(S_{2k}) \cong Z(2k; X, 1)$$

Using this, have several vectors from "planar matching" tangles

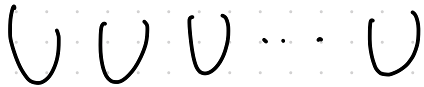
k=2



k=3



Fix your favorite two crossingless k-tangles, e.g.



T

and

S

such that $|T\rangle$ and $|S\rangle$ linearly independent. Write

$$H_k = \mathrm{span}_{\mathbb{C}}\{|T\rangle, |S\rangle\} \subseteq Z(2k; X, 1).$$

Morally: want to use $|T\rangle$ and $|S\rangle$ as computational basis states.

2 issues: orthogonal? normalized?

Of course $H_k \cong \mathbb{C}^2 = \mathrm{span}_{\mathbb{C}}\{|0\rangle, |1\rangle\}$, but choose the isomorphism so $|T\rangle$ is proportional to $|0\rangle$, i.e.

$$|0\rangle := \frac{|T\rangle}{\sqrt{\langle T|T\rangle}}$$

this will come back to haunt us!

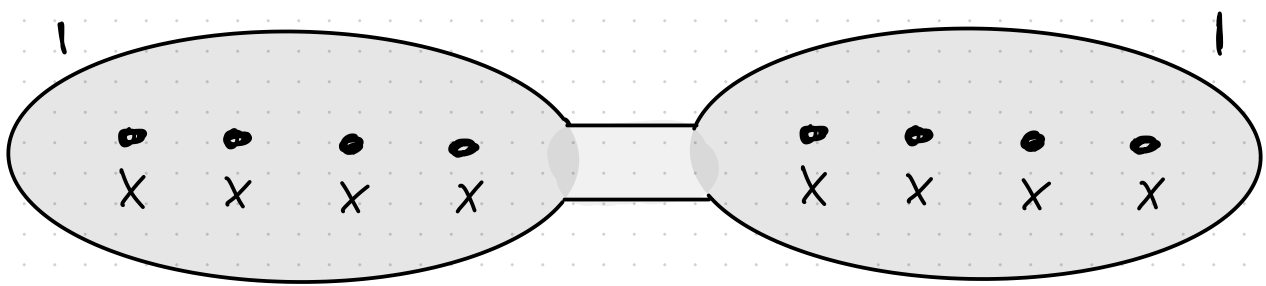
Let $|1\rangle$ be any orthonormal, e.g. Graham-Schmidt on $|T\rangle$, $|S\rangle$.

H_k with these two basis vectors will be our qubit inside of $Z(2k; X, 1)$.

Gates? Well, any dense subset of $U(\mathbb{C}^2 \otimes \mathbb{C}^2)$ will be quantum universal.

So: should consider B_{4k} acting on $Z(4k; X, 1)$, since

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong H_k \otimes H_k \subseteq Z(2k; X, 1) \otimes Z(2k; X, 1) \subseteq Z(4k; X, 1)$$



If quantum representation

$$Z: B_{4k} \longrightarrow U(Z(4k; X, 1))$$

is dense, then, in particular every "binary gate" in $U(H_k \otimes H_k)$ can be approximately implemented by a braid.

First difficulty:

$H_k \otimes H_k$ probably not an invariant subspace of B_{4k} under quantum representation $Z: B_{4k} \longrightarrow U(Z(4k; X, 1))$.

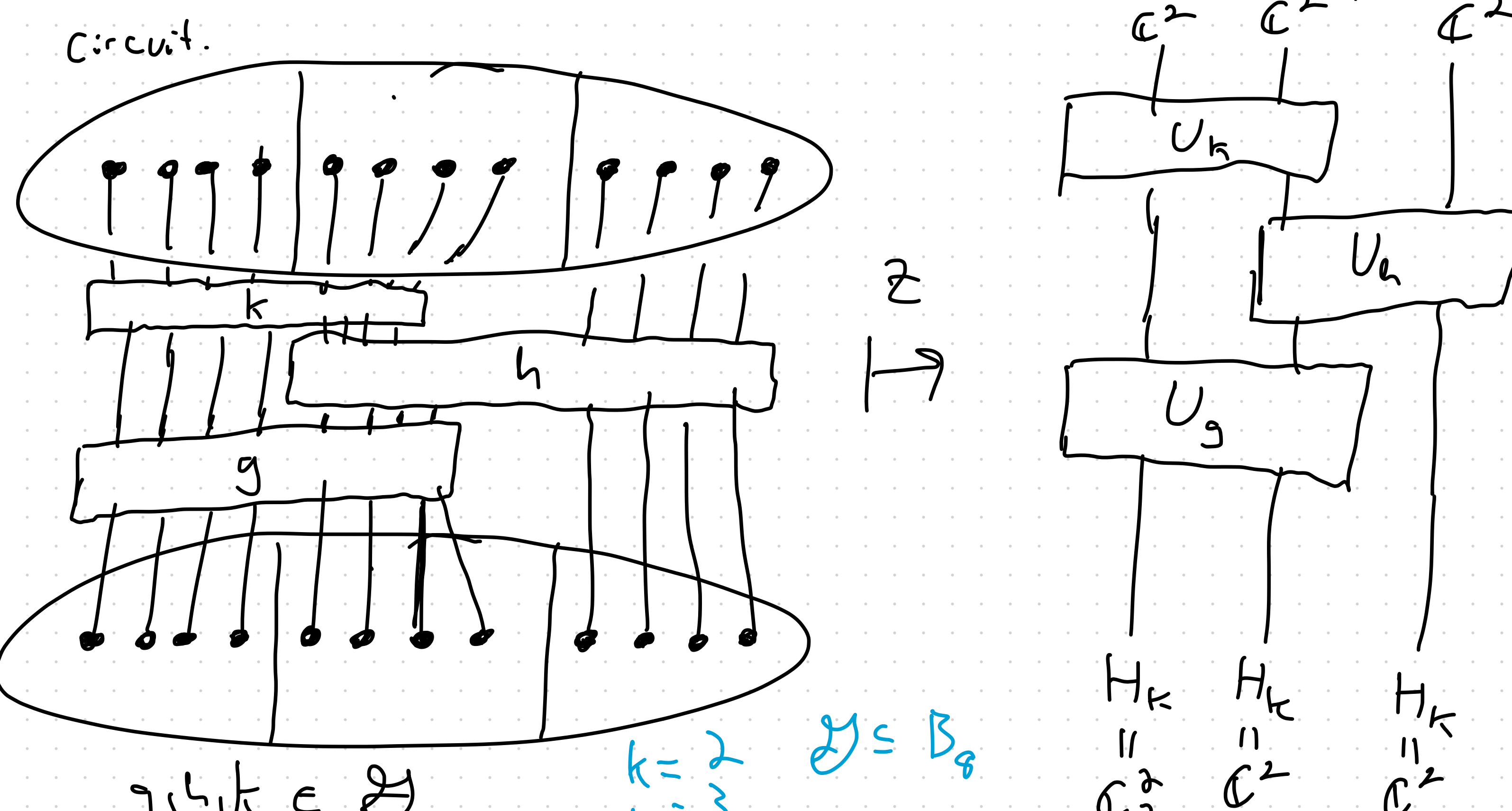
Even worse?

May not ~~be~~ be ANY braid $b \in B_{4k}$ that preserves $H_k \otimes H_k$ and acts nontrivially.

Fortunately: while of practical/engineering importance, in principle, these issues can be overcome by being careful with Solovay-Kitaev theorem.

So, we will assume there is a finite subset $\mathcal{G} \subseteq B_{4k}$ that generates a dense subgroup of $U(H_k \otimes H_k)$

Evaluating Z on braids built from $\mathcal{G}$'s acting on D_{nk} and restricting to $H_k \otimes \cdots \otimes H_k \subseteq Z(nk; X, 1)$ yields quantum circuit.



$g, h, k \in \mathcal{G}$

$k = 2$ $\quad \mathcal{G} \subseteq B_8$

$n = 3$

This shows we can simulate arbitrary quantum circuits of X particles in the TQFT $\mathcal{Z}$.

Examples? Jones-Kauffman TQFTs (proved by Freedman-Larsen-Wang)

How do topological invariants of knots/links/3-manifolds derived from & relate to this model of computation?