Meeting 2.1: Other 3-manifold encodings

I. Heegaard splittings and diagrams

II. Knots and links: stick presentations and diagrams

III. Bridge position, braid groups, and trace closures

Skipping for now: surgery presentations of 3-manifolds

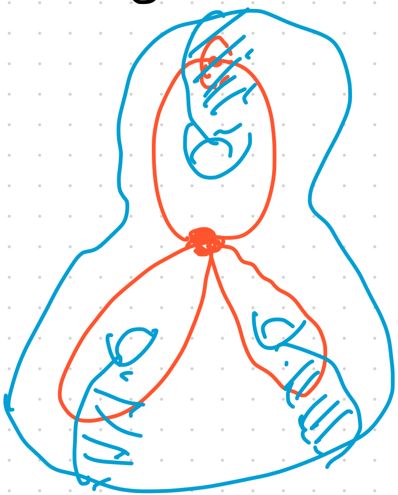
Next time: structure of 3-manifolds

Announcements: - office half hour on Wednesdays, 3:30-4:00.
w/ option to go extra half hour

- Class will end today at 3pm.

I. Heegaard splittings and diagrams

A handlebody of genus g is a 3-manifold homeomorphic to a regular neighborhood of a wedge of g circles in $\mathbb{R}^3$.



$$\bigcup_{p \in \text{Wedge}} \overline{B_\varepsilon(p)} \quad (\varepsilon \text{ small enough})$$

Let M be a (closed, orientable) 3-manifold. A Heegaard surface (or Heegaard splitting) is an embedded surface $S \subseteq M$ of some genus g such that "cutting M along S" results in two handlebodies.

"Cut M along S" means "$M - \mathring{N}(S)$."

Lemma Every (closed, orientable) 3-manifold has a Heegaard surface. More precisely, if $\mathcal{T}$ is a triangulation of a 3-manifold with t tetrahedra, then $\mathcal{T}$ has Heegaard surface of genus $= \# \mathcal{T}^1 - \# \text{Max Tree}(\mathcal{T}^1)$.

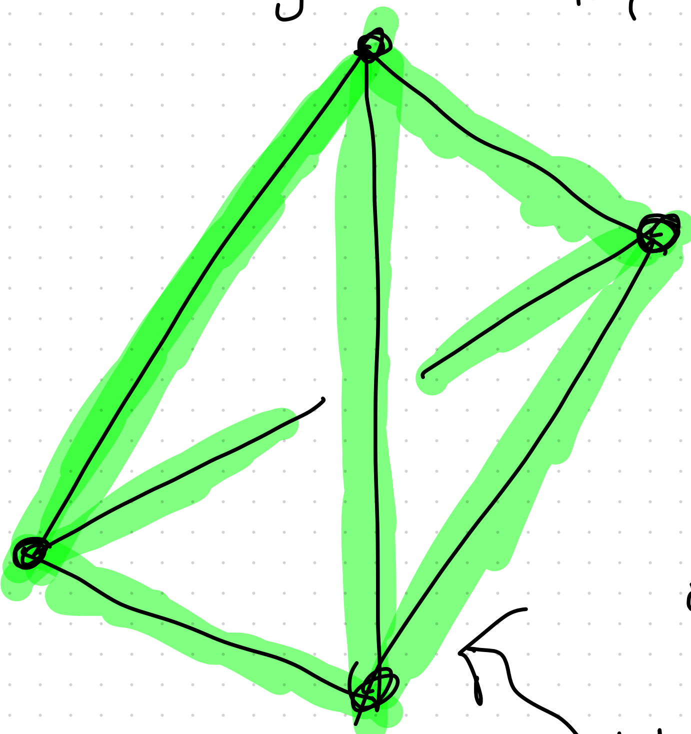
$\mathcal{T}^1$ = one skeleton

Remark: the genus will typically be way bigger than necessary. We define genus(M^3) to be the minimum genus of all Heegaard splittings of M.

genus(M^3) is an invariant.

Proof by picture

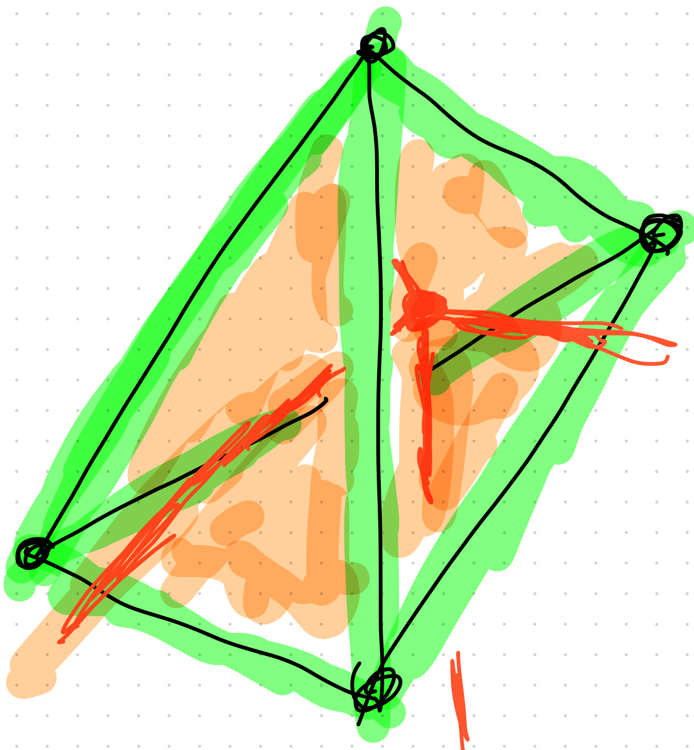
For Heegaard surface, let $S = \partial N(\Upsilon')$. We need to check that each piece of $\Upsilon - S$ is a handlebody of genus $\#\Upsilon' - \#\text{MaxTree}(\Upsilon')$.



We can contract any maximal tree in H_0 to get a homeomorphic manifold. By construction the 3-manifold is a wedge of $\#\Upsilon' - \#\text{MaxTree}(\Upsilon')$ many circles.

$H_0 = N(\Upsilon')$

Note that $H_1 = \Upsilon - \mathring{H}_0$ has boundary S.
To see that it is a 3-manifold homeomorphic to a regular nbhd of a graph, look at dual one skeleton of Υ. $\square$



We want to recover a 3-manifold from a surface (which will be the Heegaard splitting) + finite amount of extra data

Def'n Call a collection of simple closed curves $c_1, c_2, \dots, c_g$ on a genus g surface S a <u>complete disk system</u> if:

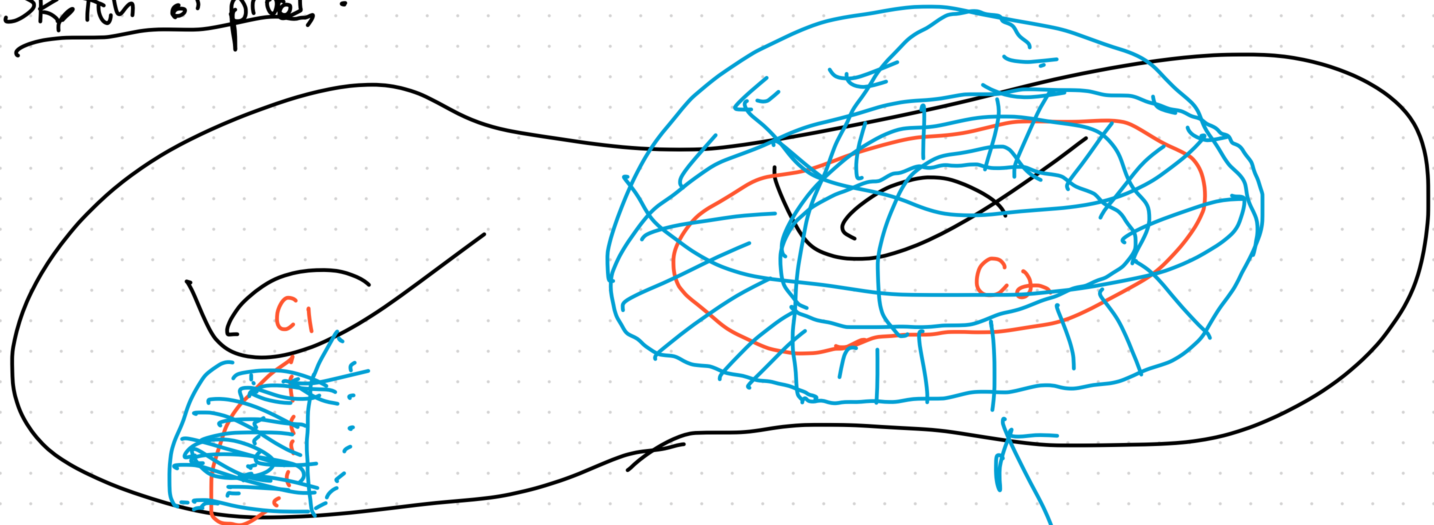
i) c_i's pairwise disjoint

ii) $S - \bigcup_{i=1}^{g} c_i$ is connected.

Theorem Let S be a genus g surface with a complete disk system $c_1, \dots, c_g$. Then, up to (orientation-preserving) homeomorphism rel S, there is a unique handlebody H with $\partial H = S$ such that each c_i bounds an embedded disk.

Sketch of proof:

$S \times [-1,0]$



glue $D^2 \times [-1,1]$
so $D^2 \times \{0\}$
is identified w/ c_1

Similarly over here

$$N = \bigcup_{i=1}^{g} N(c_i)$$

$$D = \bigcup_{i=1}^{g} D^2 \times \{-1,+1\}$$

Then $D \cup (S \setminus N)$ is a 2-sphere.

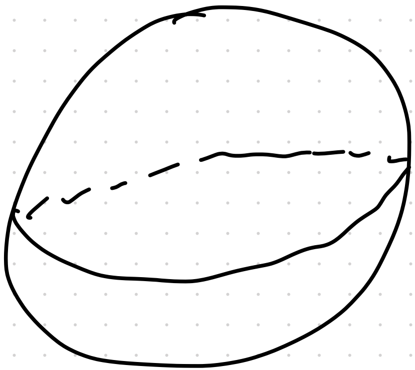
Up to orientation preserving isotopy, there is exactly one way to glue a 3-ball to $D \cup (S \setminus N)$.

$\square$

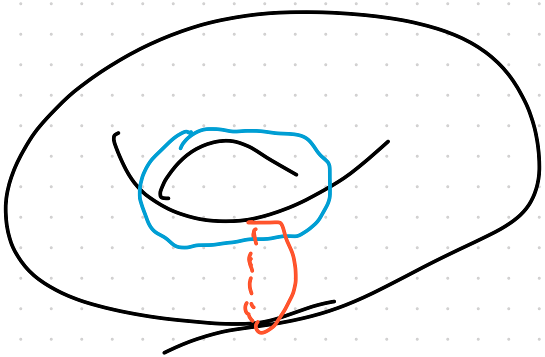
Corollary Every closed orientable 3-manifold can be presented as a Heegaard diagram, which consists of a surface S of some genus g, together w/ two complete disk systems on S

In fact, S can be triangulated, and each curve in the disk systems is a normal curve.

Example



$\rightsquigarrow S^3$



$\to S^3$