Meeting 2.2: More on 3-manifold encodings

I. Normal curves and Heegaard diagrams

II. Knots and links: stick presentations and diagrams

Next week: 1. Complexity theory (not structure of 3-manifolds)

2. Example problems for 3-manifolds and their complexities.

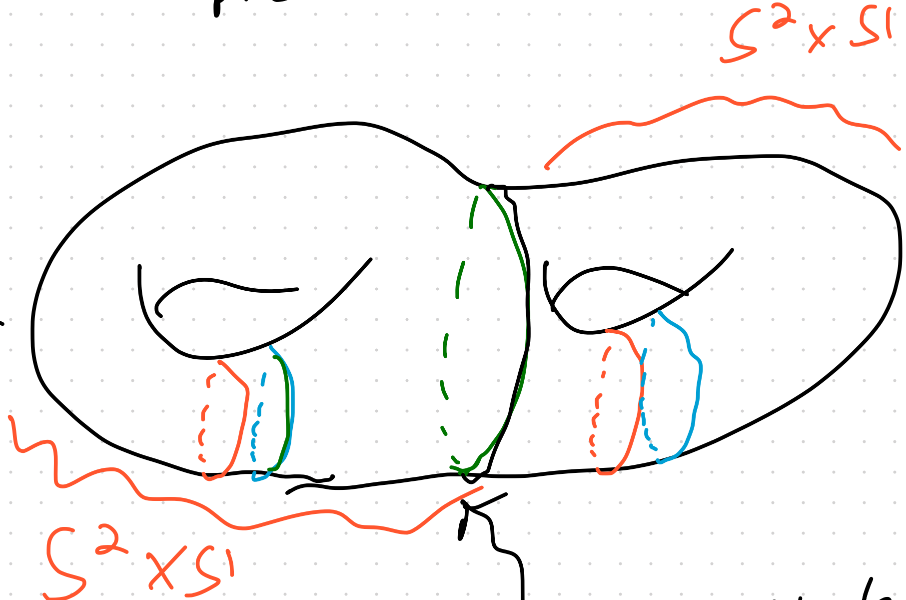
I. Normal curves and Heegaard diagrams

Last time:

Corollary Every closed orientable 3-manifold can be presented as a Heegaard diagram, which consists of a surface S of some genus g, together w/ two complete disk systems on S

In fact, S can be triangulated, and each curve in the disk systems can be made a normal curve.

More examples:

$M =$ 

$\cong S^2 \times S^1 \# S^2 \times S^1$

green curve bounds disk on both sides

Corresponds to a 2-sphere embedded in M

Connect sums of manifolds:

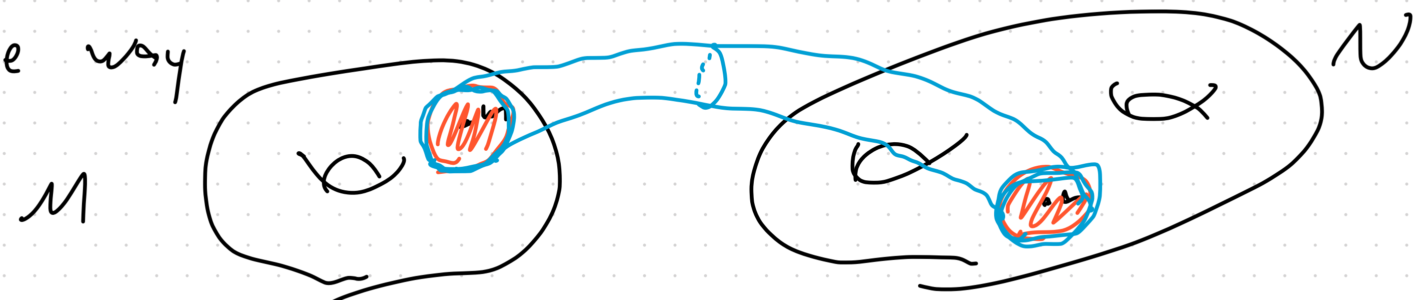
Let M, N be two orientable 3-folds (connected).

Pick $m \in M$, $n \in N$. Let

$$M' = M - \overline{B_\varepsilon(m)}, \quad N' = N - \overline{B_\varepsilon(n)}$$

M' and N' each has a new S^2 boundary component.

B/c orientable, we can identify two copies of S^2 in a unique way

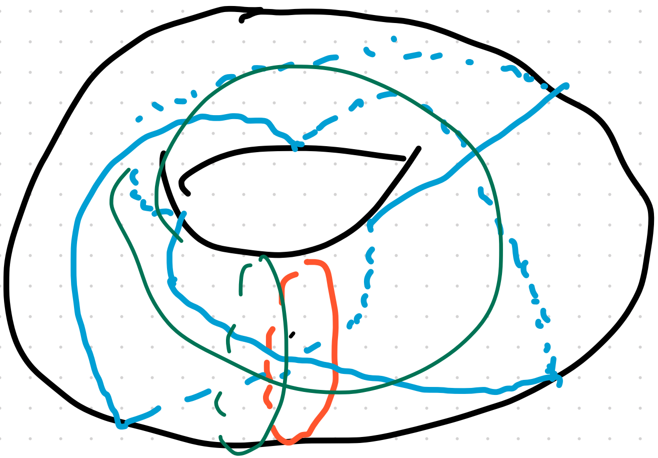


Let $M \# N$ be

$$M' \sqcup N'$$
$$\partial M' \xrightarrow{\sim} \partial N'$$

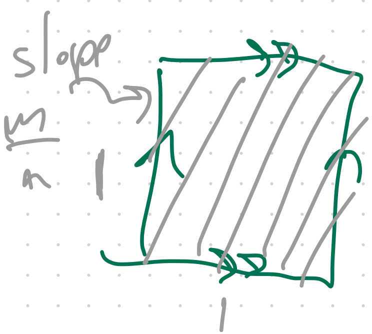
Conversely: a 3-manifold L is a connect sum precisely when there exists an embedded 2-sphere $S^2 \subset L$ s.t. neither component of $L - S^2$ is a 3-ball.

More examples: every manifold with a genus one splitting is called a lens space:

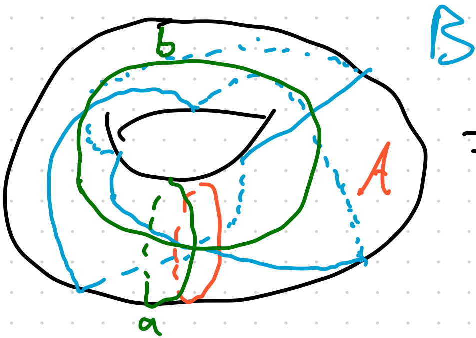


$= L(3,2)$

- Can build $L(m,n)$ for any m,n.
- Classified up to homeomorphism by
- Non-homeomorphic lens spaces can be homotopy equivalent.



Let's compute something:



$= L(3,2)$

$L = A \cup B \qquad A \cap B = S^1 \times S^1, \quad A \cong B \cong S^1 \times D^2$

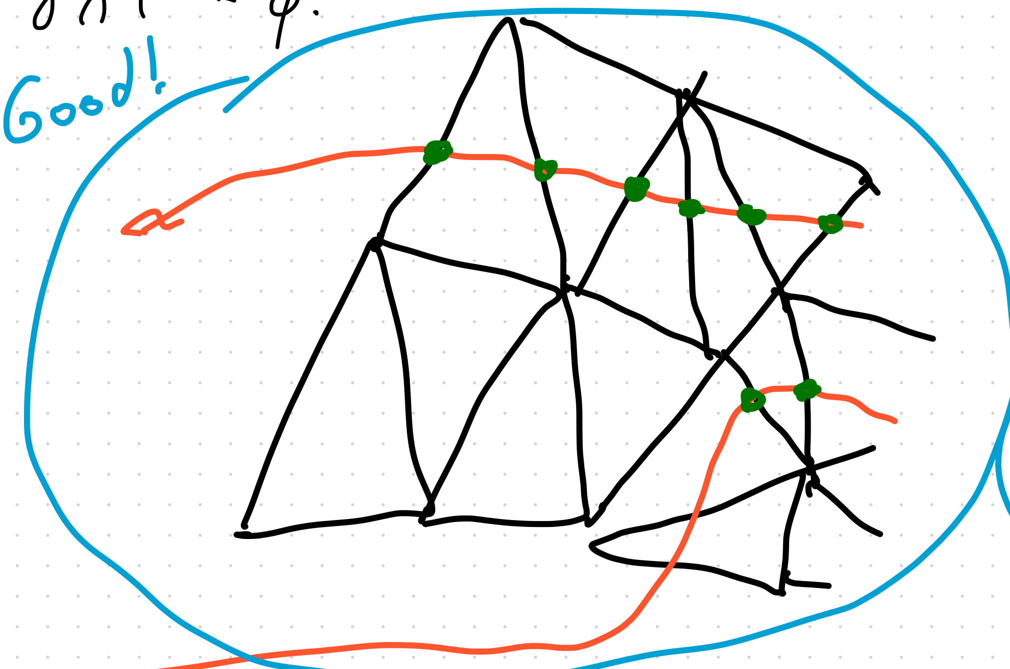
Mayer-Vietoris sequence

$$\begin{tikzcd}
H_1(A \cap B) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & H_1(A) \oplus H_1(B) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & H_1(L) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & 0 \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}"] \arrow[d, phantom, "=" rotate=-90] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z}/3 & \\
\langle a \rangle \oplus \langle b \rangle & & &
\end{tikzcd}$$

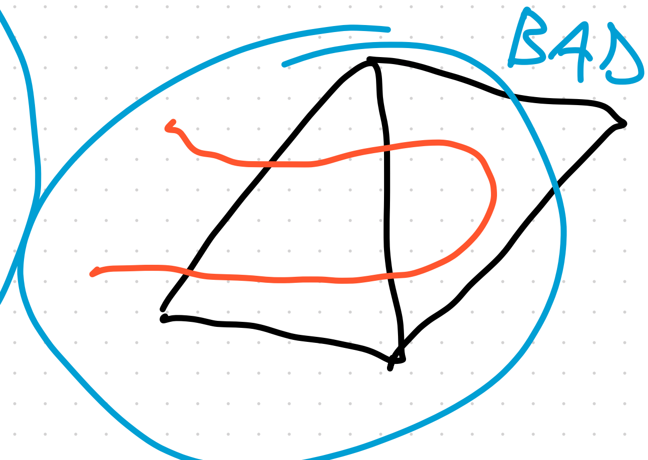
$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

Normal curves

Let $\mathcal{T}$ be a triangulated surface. A curve γ is <u>normal</u> (wrt $\mathcal{T}$) if every segment in $\gamma - \mathcal{T}^1$ has its endpoints on distinct edges of $\mathcal{T}^1$, and $\gamma \cap \mathcal{T}^0 = \emptyset$.



Ruling out curves that "backtrack"

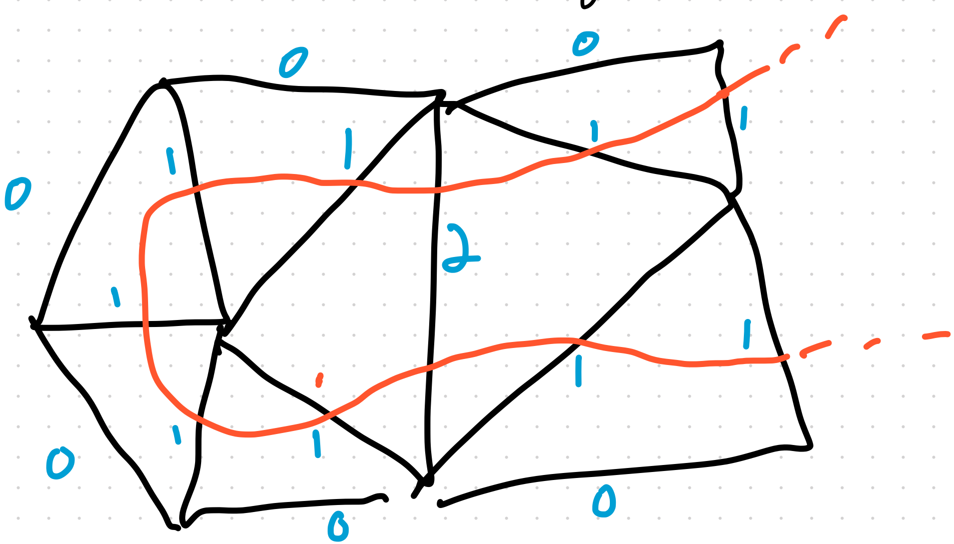


Normal curves

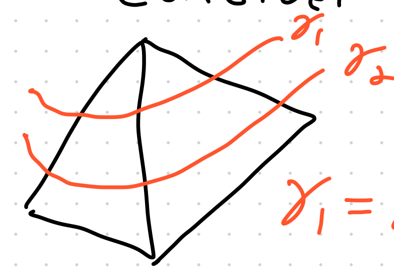
Up to isotopy in $\Upsilon - \Upsilon^0$, a normal curve is determined by a vector of edge intersection counts:

$$\nu_\gamma : \mathrm{Edges}(\Upsilon) \to \mathbb{Z}_{\geq 0}$$

$$\nu_\gamma(E) = \# \gamma \cap E.$$



Consider



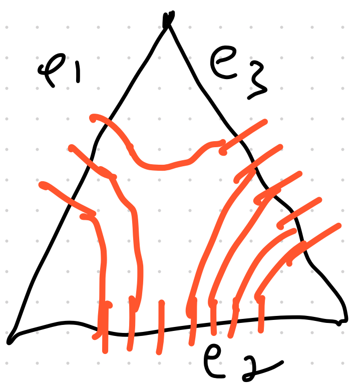
$\gamma_1 = \gamma_2$

Normal curves

The set of (isotopy classes rel $\mathcal{T}^0$ of) normal curves is a polyhedral cone in $\mathbb{Z}^{\text{Edges}}$.

Claim: a vector $v \in \mathbb{Z}^{\text{edges}}_{\geq 0}$ determines a normal curve if and only if, for each triangle $T \in \mathcal{T}$, the three corresponding entries of v satisfy triangle inequalities

3 5
7



Meeting 2.2: More on 3-manifold encodings

I. Normal curves and Heegaard diagrams

II. Knots and links: stick presentations and diagrams

III. Bridge position, braid groups, and trace closures

Next week: 1. Complexity theory (not structure of 3-manifolds)

2. Example problems for 3-manifolds and their complexities.

Warning

Triangulation $\longrightarrow$ Heegaard diagram easy, but converse is usually expensive.

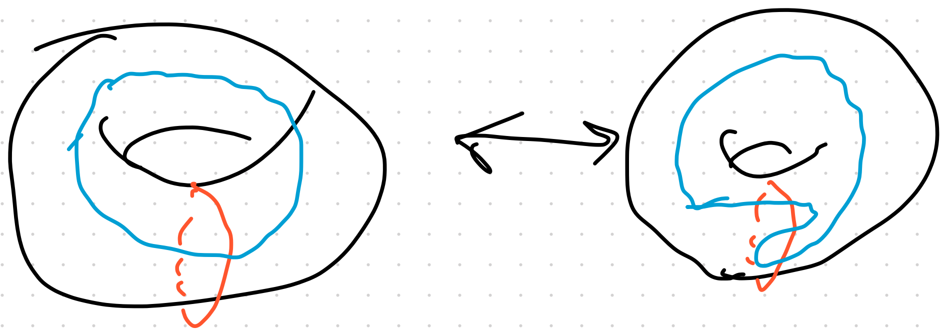
Problem: a normal curve vector $v \in \mathbb{Z}_{\geq 0}^{\text{Edges}}$ can encode a curve that is exponentially long in the size of v.

Takeaway: Heegaard diagram is "highly compressed"

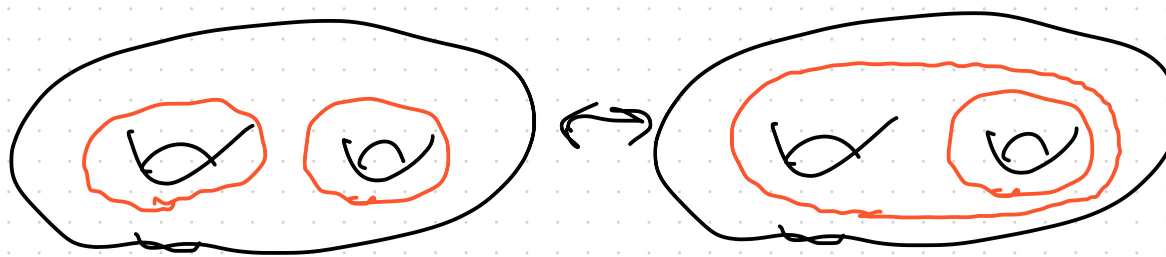
Deciding homeomorphism from Heegaard diagrams

Reidemeister - Singer Two Heegaard diagrams represent homeomorphic manifolds if and only if they can be identified by a sequence of elementary operations

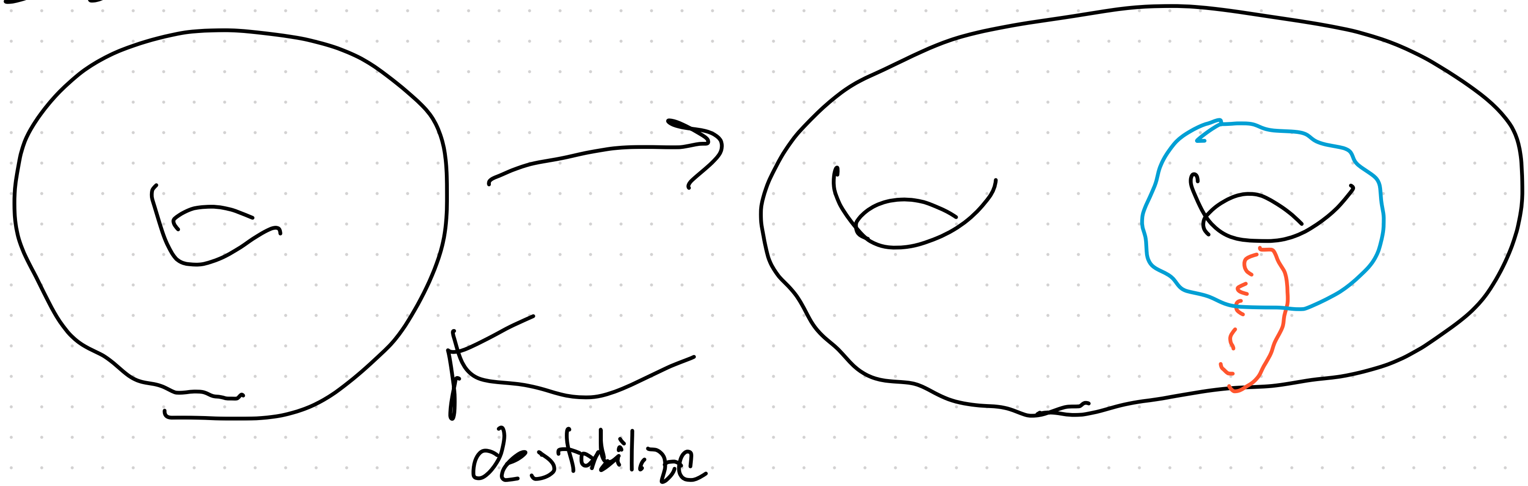
1. Isotopy



2. Handle slide



3. Stabilization



II. Knots and links: stick presentations and diagrams

A *knot* is a continuous injection (embedding)

$$K : S^1 \to \mathbb{R}^3 \quad (\text{or } S^3).$$

Often conflate a knot K with its image.

When are two knots equivalent?

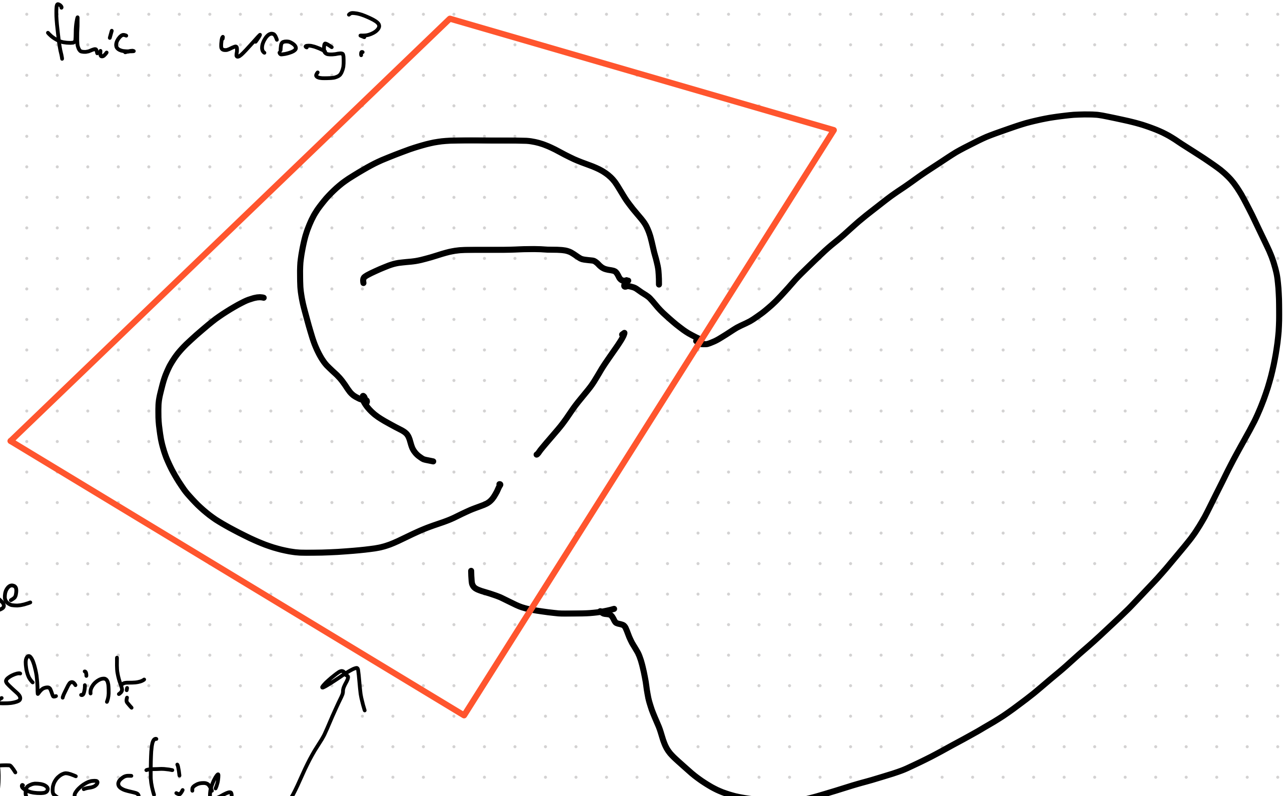
One wrong answer: isotopy.

Recall, two maps $K, L : S^1 \to \mathbb{R}^3$ are *isotopic* if there is a continuous function

$$H : S^1 \times [0,1] \to \mathbb{R}^3$$

such that $H|_{S^1 \times \{0\}} = K$ and $H|_{S^1 \times \{1\}} = L$, and $H(-,t)$ for any fixed t is an embedding.

Why is this wrong?



Can use H to shrink the interesting part

Isotopies can make knots trivial!

One correct definition: ambient isotopy

K and L are ambient isotopic if there exists

$$H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$$

such that

i) $H(-,t)$ is a homeomorphism of $\mathbb{R}^3$

ii) $H(-,0) = id_{\mathbb{R}^3}$

iii) $H(K(x),1) = L(x)$ for all $x \in S^1$.

Another correct definition: Homeomorphism!

Say K and L are homeomorphic if there exists

$$h : \mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^3$$

such that $h(K(x)) = L(x)$ for all $x \in S^1$.

Two definitions ALMOST identical:

every orientation preserving homeo of $\mathbb{R}^3$ (or S^3) is isotopic to the identity.

Ex:  vs  (left-handed vs right-handed trefoils)