Meeting 3.1

I. Knots and links: stick presentations and diagrams

II. Bridge position, braid groups, and trace closures

III. Decision problems, counting problems, and computability

I. Knots and links: stick presentations and diagrams

Recall, a knot is a cts. embedding

$$K: S^1 \to S^3$$

A knot is tame if it comes from restricting a embedding

$$N(K): S^1 \times D^2 \to S^3$$

via $N(K)|_{S^1 \times \{0\}}$.

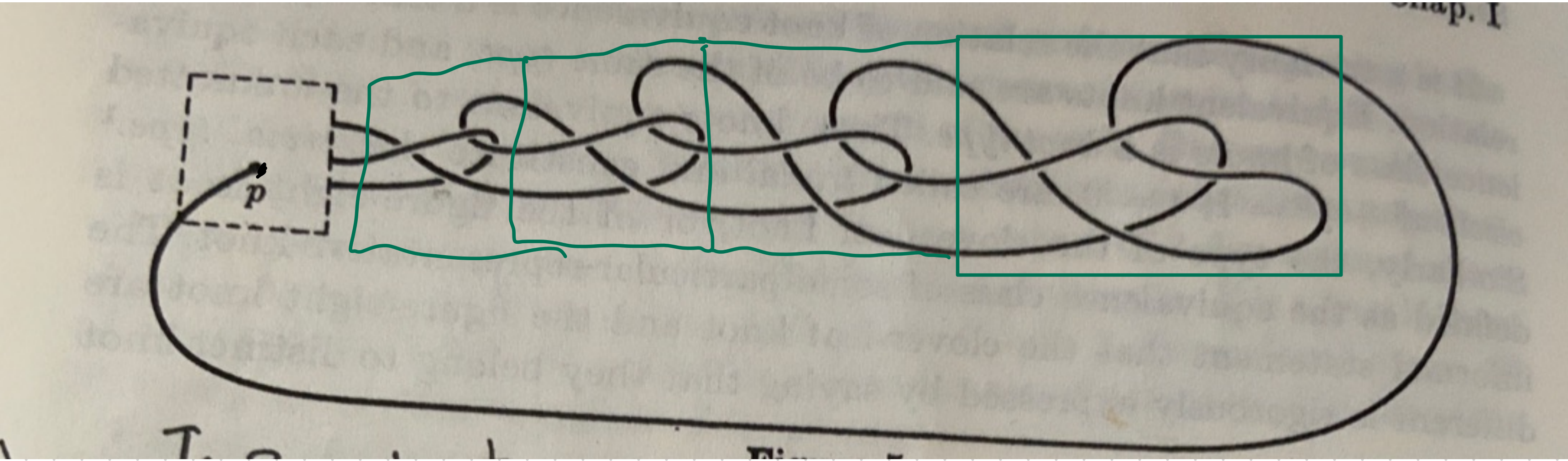
"Tame" = "Has a tubular neighborhood"

A knot that is not tame is called wild.

Any knot K that "extends across a disk" is an unknot.

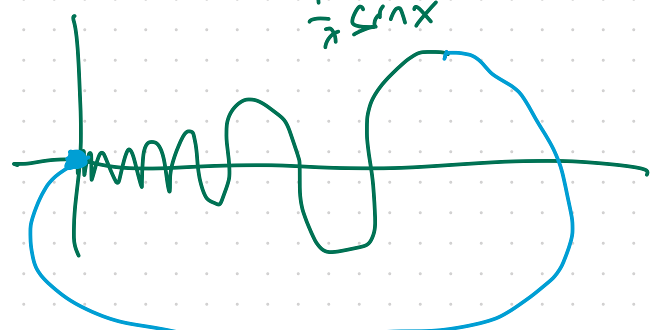
That is if there exists an embedding $T: D^2 \to S^3$ such that $T|_{\partial D^2} = T|_{S^1} = K$.

WILD KNOT

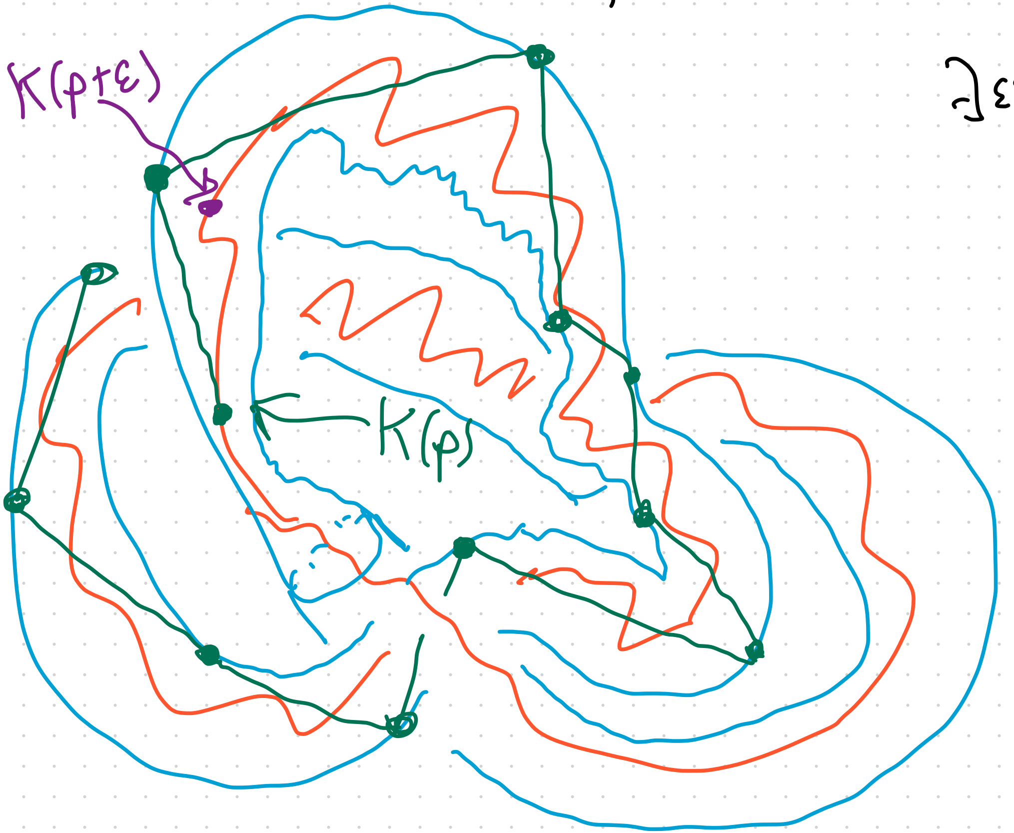


found by Fox.

Why not tame?



Tame knots are always isotopic to PL knots



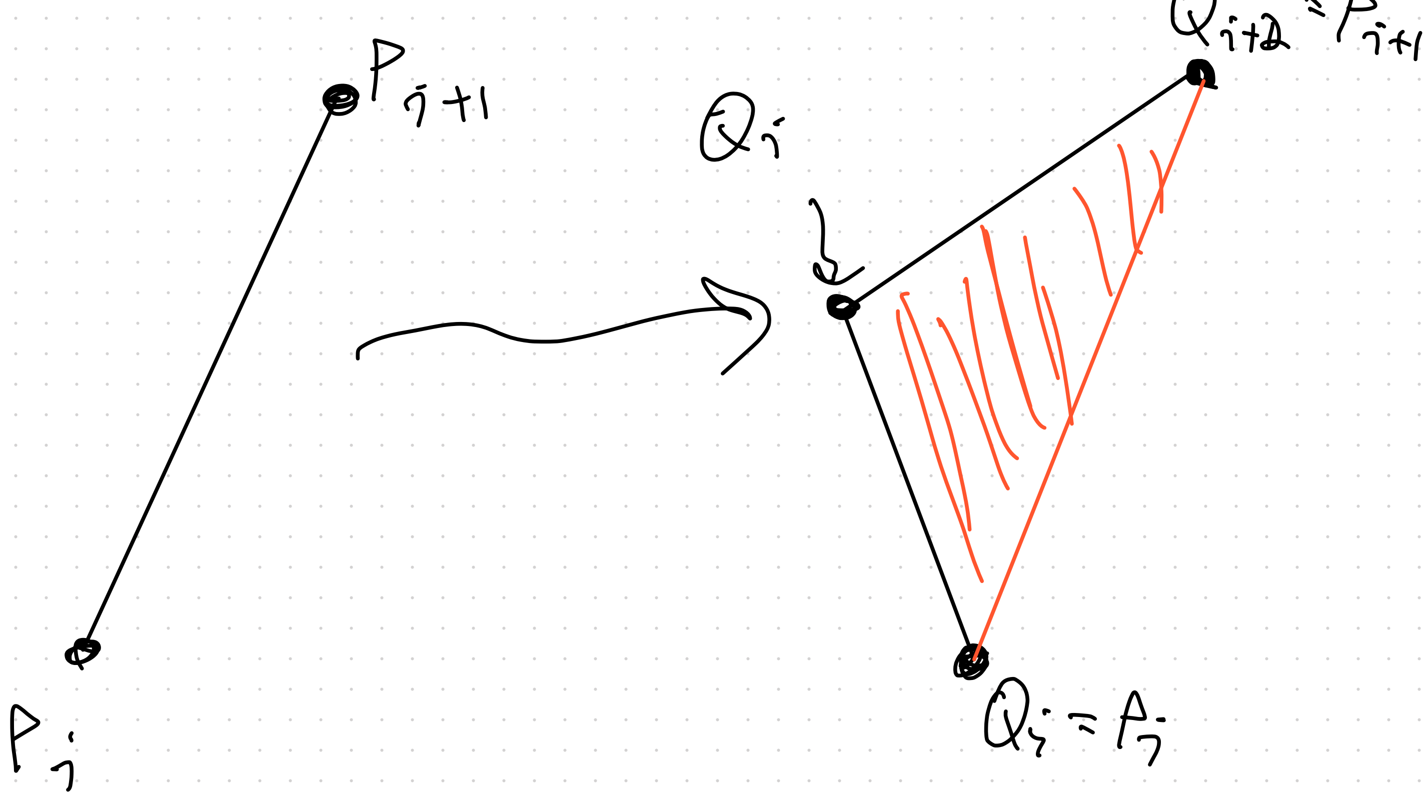
$\exists \varepsilon > 0 : \forall p \in S^1,$

$$[K(p), K(p+\varepsilon)] \subseteq \operatorname{Image}(N)$$

Stick presentations and triangle moves

A stick presentation of a knot is a sequence of points $P_0, P_1, \dots, P_L = P_0 \in \mathbb{Z}^3 \subseteq \mathbb{R}^3 \subseteq S^3$ so that for all $i, j = 0, \dots, L-1$, the line segments $P_i P_{i+1}$ and $P_j P_{j+1}$ have disjoint interiors and $P_i \neq P_j$.

A triangle move is an elementary isotopy between two stick presentations:



If Q is any point in $\mathbb{R}^3$ such that the triangle $P_i Q P_{i+1}$ is disjoint from all of the other sticks, then two stick presentations related by a triangle move represent isotopic knots

<u>Theorem</u> Two stick presentations represent equivalent knots (ambient isotopic) if and only if they are related by a sequence of triangle moves.

Diagrams and Reidemeister moves

A diagram of a knot is an embedded planar graph with extra information at vertices to encode crossing information. This planar graph should come from a regular projection of a knot K in $\mathbb{R}^3$ onto a plane.

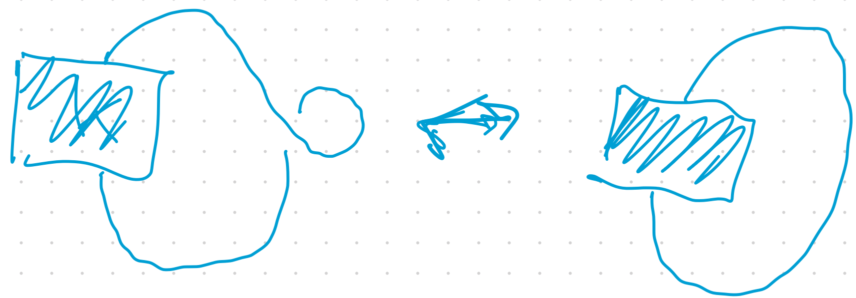
- Require preimage of every point to have at most 2 points
- Also don't allow

- We're forcing all crossing singularities to be transverse

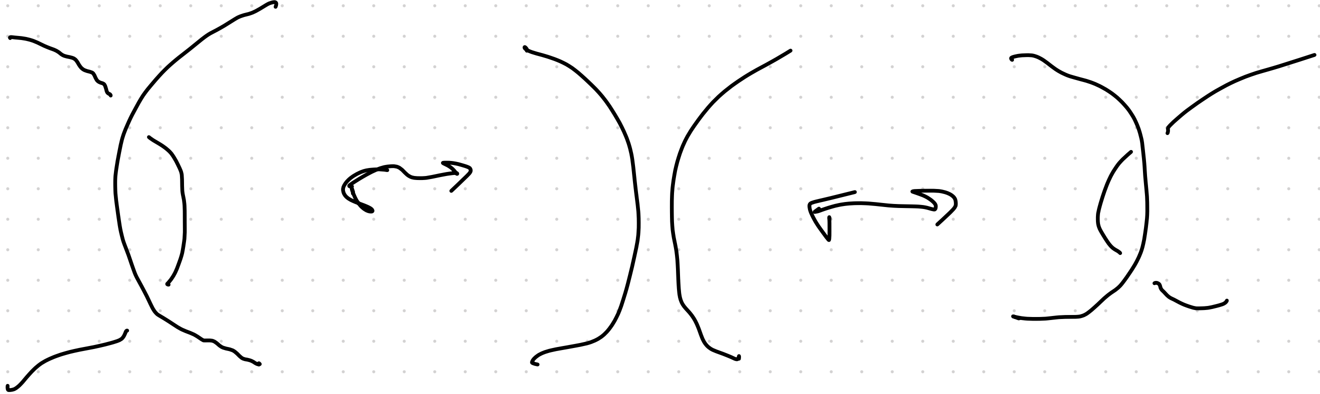
Diagrams and Reidemeister moves

Thm Two knot diagrams represent equivalent knots iff they are related a sequence of Reidemeister moves:

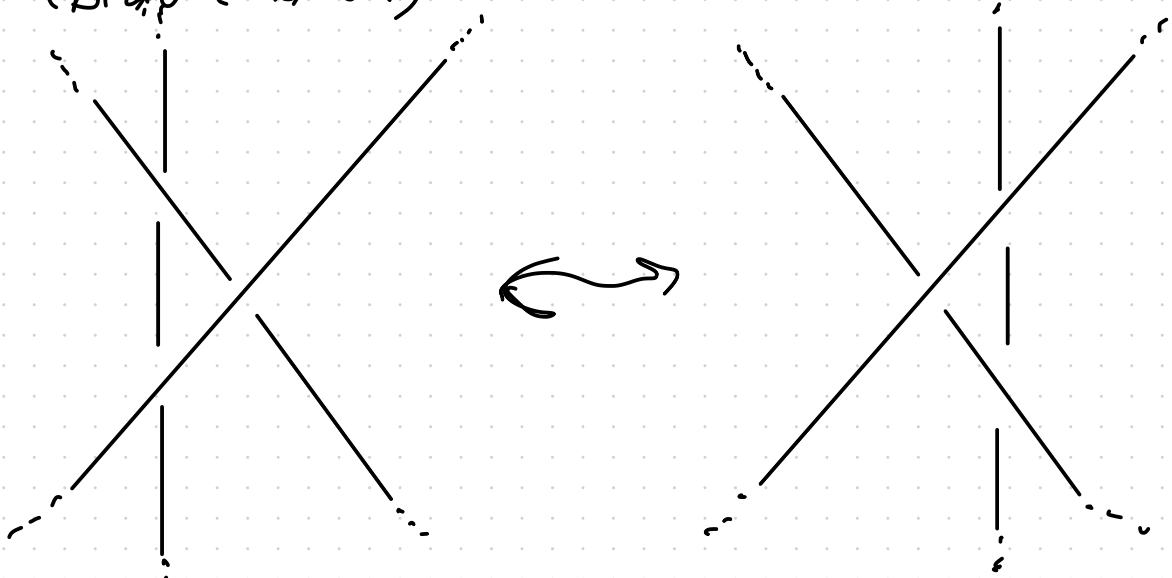
Type I



Type II



Type III (braid relation)

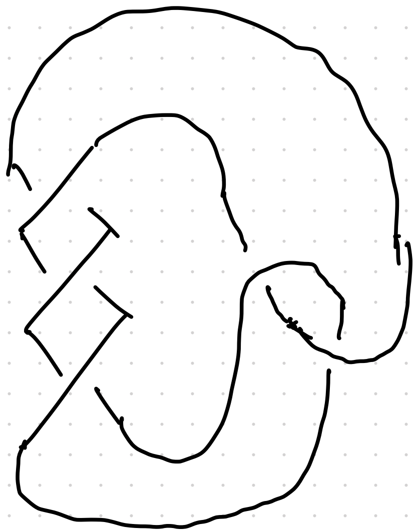


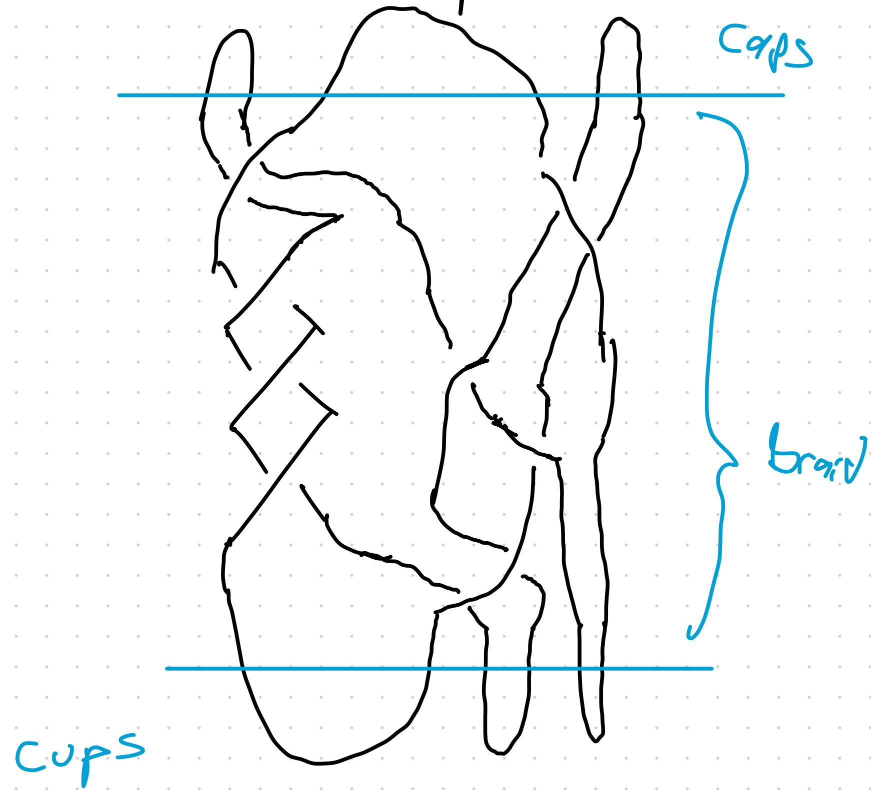
II. Bridge position, braid groups, and trace closures

A knot diagram is in <u>bridge position</u> iff, when considered as a subset of xy-plane, all of the maxima occur at same height, and all of the minima " " " " .

Proposition Given any knot diagram, we can easily find an equivalent diagram in bridge position.

Proof:





Artin's braid group(s)

B_n is the braid group on n strands, which is presented via

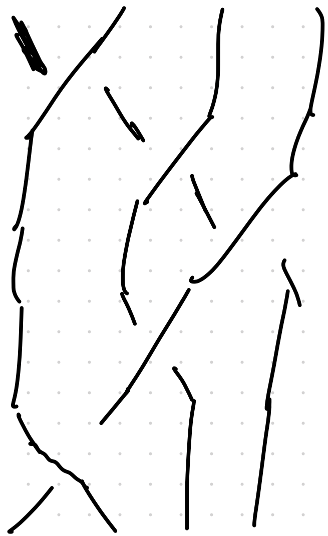
$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ \forall i \end{array} \right\rangle$$

We can interpret a string of σ_i's as a picture of a braid; the relations ensure that isotopic braids are considered equal elements of the group.

Ex Consider $\sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_1^{-1}$ in B_4

Convention: "Right to left" = "Bottom to top"

σ_1
σ_2
σ_3
σ_2
σ_1^{-1}



$=$

σ_1
σ_3
σ_2
σ_3
σ_1^{-1}

literal equality in group corresponds to isotopy of braid diagrams

Given any word in the generators of B_n we can draw a braid diagram.

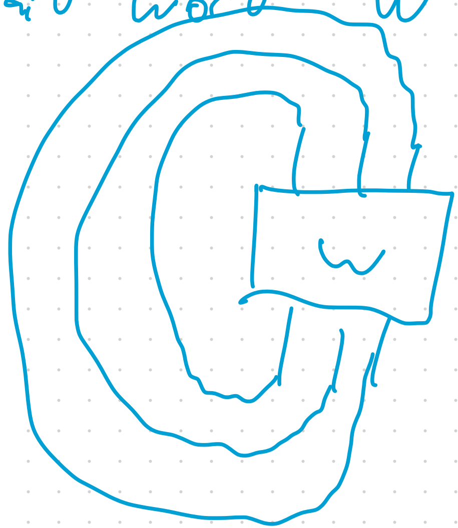
Furthermore, if $n=2k$ is even, the following three pieces of data give us a diagram of a knot in bridge position:

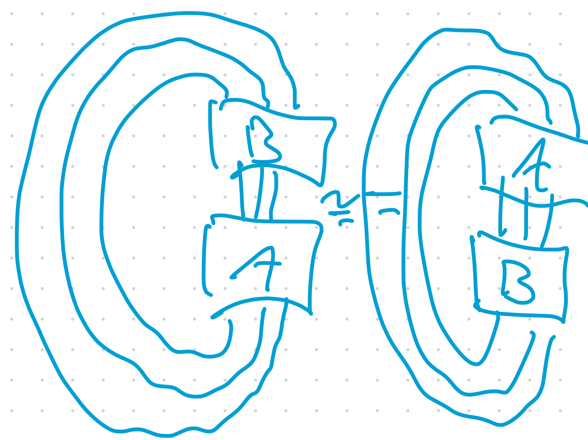
1. Cups (a planar matching of the $2k$ strands)
2. Caps (ditto)
3. A word in generators of B_{2k}.

Warning: Previous recipe might yield a link diagram instead of a knot diagram.

Alternative construction: Trace closure.

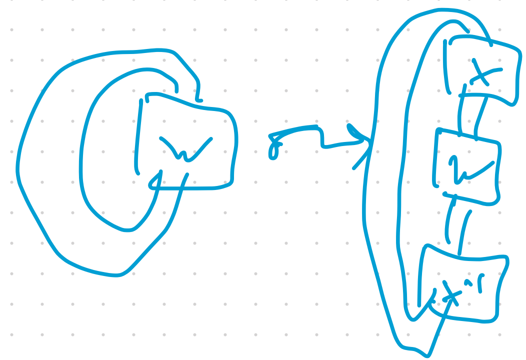
Given braid word $w \in B_n$, do this:





Two Markov closures of braids represent equivalent links when they are related by a sequence of two types of moves:

1. Conjugacy: $w \rightsquigarrow x w x^{-1}$



2. Stabilization: