Meeting 4.2: The complexity of unknot recognition

I. In NP via normal surface theory (Hass-Lagarias-Pippenger, after Haken)

II. In NP via Reidemeister moves (Lackenby)

III. In coNP, modulo GRH (Kuperberg)

IV. In coNP (Lackenby)

Next week: quantum mechanics and quantum computing

Problem: Unknot Recognition

Instance: Knot diagram K

Question: Is K the unknot?

K is an unknot if:

1. it's equivalent under a sequence of Reidemeister moves to a diagram w/o crossings
2. if K bounds an embedded disk
3. $\pi_1(S^3 - K) \cong \mathbb{Z}$, infinite cyclic group

I. In NP via normal surface theory (Hass-Lagarias-Pippenger)

The Computational Complexity of Knot and Link Problems

Journal of the ACM, Vol. 46, No. 2, March 1999, pp. 185–211.

JOEL HASS

University of California, Davis, Davis, California

JEFFREY C. LAGARIAS

AT&T Laboratories, Florham Park, New Jersey

AND

NICHOLAS PIPPENGER

University of British Columbia, Vancouver, BC, Canada

In this paper, we consider knots and links as represented by link diagrams and take the crossing number as the measure of input size. We can now formulate the computational problem of recognizing unknotted polygons as follows:

Problem: UNKNOTTING PROBLEM
Instance: A link diagram $\mathfrak{D}$
Question: Is $\mathfrak{D}$ a knot diagram that represents the trivial knot?

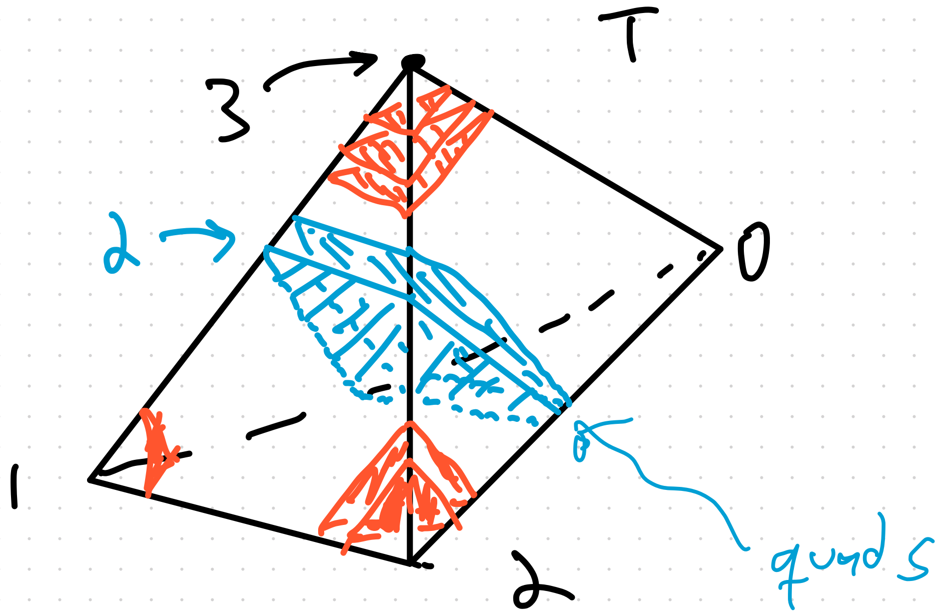
See Welsh [1993a–1993c] for more information on this problem. The main result of this paper is the following:

THEOREM 1.1 *The* UNKNOTTING PROBLEM *is in* **NP**.

The UNKNOTTING PROBLEM was shown to be decidable by Haken [1961]; the result was announced in 1954, and the proof published in 1961. From then until now, we know of no strengthening of Haken's decision procedure to give an explicit complexity bound. We present such a bound in Theorem 8.1.

Given $\mathcal{T}$, triangulated 3-manifold, we can describe "normal surfaces" using certain vectors $\mathbb{Z}^{7t}$, where t is # tetrahedra in $\mathcal{T}$.

Inside each tetrahedron T, we have 7 types of elementary disks:

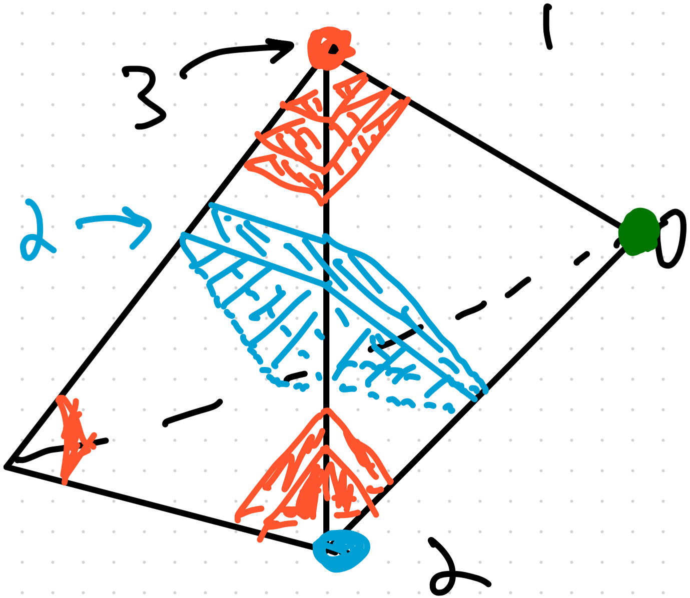


A vector $v \in \mathbb{Z}^{7t}$ determines a normal surface if

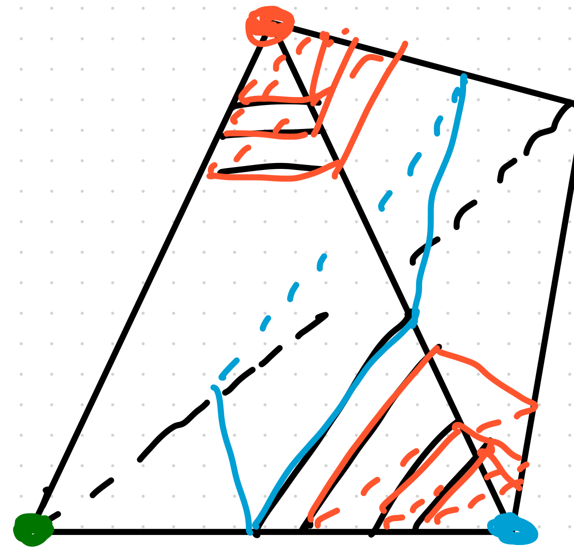
- $v_i \geq 0$ for each $i = 1, \dots, 7t$
- "matching condition" for each pair of faces that are glued together of the form

$$v_{i_1} + v_{i_2} = v_{k_1} + v_{k_2}$$

- "quad condition": for each tetrahedron, see at most one quad type



1
3
2
0
2



Basic idea: effectivizing Haken's normal surface theory

1. Given K, a knot diagram with n crossings, can build triangulation of $M_K := S^3 - N(K)$ in time $O(n \log n)$ with $t = O(n)$ tetrahedra, in standard way.

Certificate: A vector $v \in \mathbb{Z}^{7t}$, and a list of $7t-1$ linear constraints, with $v_i \leq 2^{7t-1}$

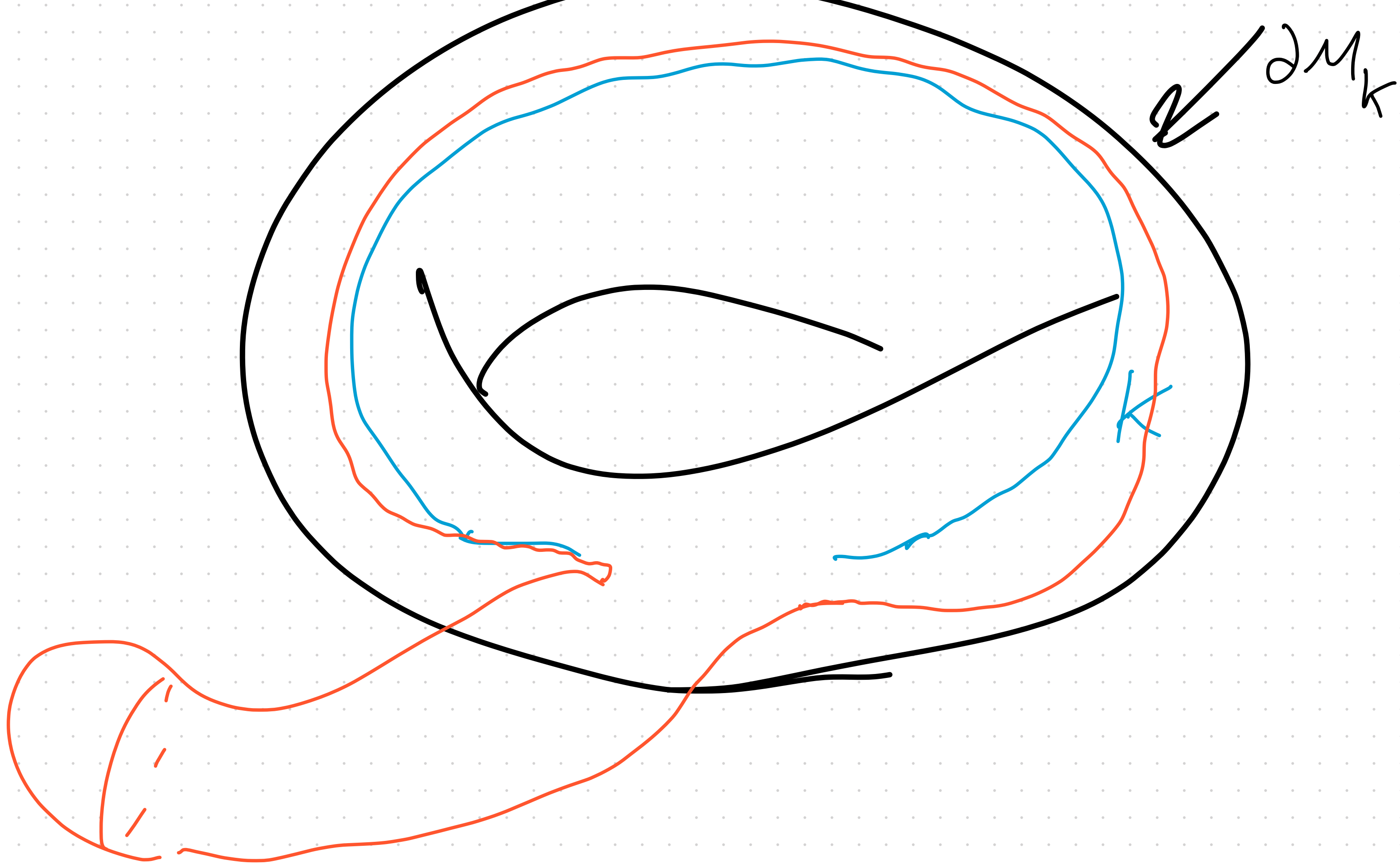
2. Test that v encodes a normal surface by verifying it satisfies:

- "matching equations"
- positivity: $v_i \geq 0 \quad \forall i = 1, \dots, 7t$
- "quad conditions" ← Nonlinear, but easy to check

Define Haken's normal cone C_M in $\mathbb{R}^{7t}$.

3. Verify the $7t-1$ linear constraints are independent, v satisfies them, and $\gcd(v_1, \dots, v_{7t}) = 1$. This shows v is a "vertex minimal" integer point in C_M.
4. Vertex minimality ensures v represents a connected normal surface. Now check v represents a disk (compute χ) and has correct boundary on $\partial M_K = S^1 \times S^1$ (check $[\partial v] = (0,1) \in H_1(S^1 \times S^1)$).

Procedure works by theorem of Jaco-Tollefson saying if K is unknot, it has a vertex minimal disk. HLP showed there exists one w/ $v_i \leq 2^{7t-1}$.



∂M_k
K

II. In NP via Reidemeister moves (Lackenby)

Annals of Mathematics **182** (2015), 491–564
http://dx.doi.org/10.4007/annals.2015.182.2.3

A polynomial upper bound on Reidemeister moves

By Marc Lackenby

Theorem 1.1. *Let D be a diagram of the unknot with c crossings. Then there is a sequence of at most $(236\,c)^{11}$ Reidemeister moves that transforms D into the trivial diagram. Moreover, every diagram in this sequence has at most $(7\,c)^2$ crossings.*

Subtle warning:

Theorem 1. *Given an unknot diagram D and an integer k, deciding if D can be untangled using at most k Reidemeister moves is* **NP***-complete.*

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The unbearable hardness of unknotting ☆

Arnaud de Mesmay [a,*], Yo'av Rieck [b,*], Eric Sedgwick [c,*], Martin Tancer [d,*]

[a] *Université Paris-Est, LIGM, CNRS, ENPC, ESIEE Paris, UPEM, Marne-la-Vallée, France*
[b] *Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701, USA*
[c] *School of Computing, DePaul University, 243 S. Wabash Ave, Chicago, IL 60604, USA*
[d] *Department of Applied Mathematics, Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic*

Moral:

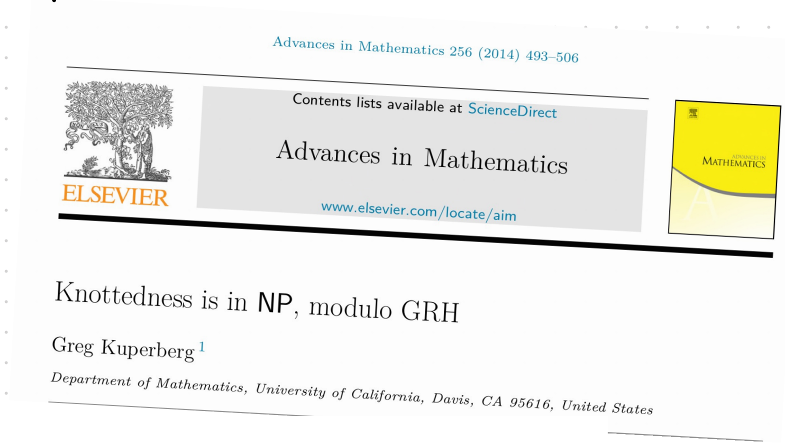
Optimal unknotting via Reidemeister moves is NP-hard.

III. In coNP, modulo GRH (Kuperberg)

Opposite question (Knottedness detection)

Instance: knot diagram K

Question: Is K NOT the unknot?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Knottedness is in NP, modulo GRH

Greg Kuperberg [1]

Department of Mathematics, University of California, Davis, CA 95616, United States

Theorem 1.1. *Let $K \subset S^3$ be a knot described by a knot diagram, a generalized triangulation, or an incomplete Heegaard diagram. Then the assertion that K is knotted is in* NP, *assuming the generalized Riemann hypothesis* (*GRH*).

Together with Hass–Lagarias–Pippenger, we can restate the result as

$$\text{Unknottedness} \in \mathsf{NP} \cap \mathsf{coNP},$$

Problems in here are believed not to be NP-hard.

Our proof of Theorem 1.1 quickly follows from major results of others. Kronheimer and Mrowka [20] showed that if K is a non-trivial knot, then there is a non-commutative representation of

of

$\mathbb{Z}/p$ i.e. image of $\rho_\mathbb{C}$ is non abelian

$$\rho_\mathbb{C} : \pi_1(S^3 \setminus K) \to \mathrm{SU}(2) \subset \mathrm{SL}(2, \mathbb{C}).$$

Then, simply because the equations for the representation are algebraic, the complex numbers can be replaced by a finite field $\mathbb{Z}/p$. Koiran [19] showed that if a polynomial-length set of algebraic equations has a complex solution, and if GRH is true, then there is a suitable prime p with only polynomially many digits. Thus, the certificate is a prime p and a 2×2 matrix over $\mathbb{Z}/p$ for each generator of the knot group. The verifier must check that the generator matrices satisfy the relations of the knot group; and that they do not all commute, or in the Wirtinger presentation, that they are not all equal. This confirms that K cannot be the unknot.

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Knottedness is in NP, modulo GRH

Greg Kuperberg [1]

Department of Mathematics, University of California, Davis, CA 95616, United States

IV. In coNP (Lackenby)

THE EFFICIENT CERTIFICATION OF KNOTTEDNESS AND THURSTON NORM

MARC LACKENBY

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