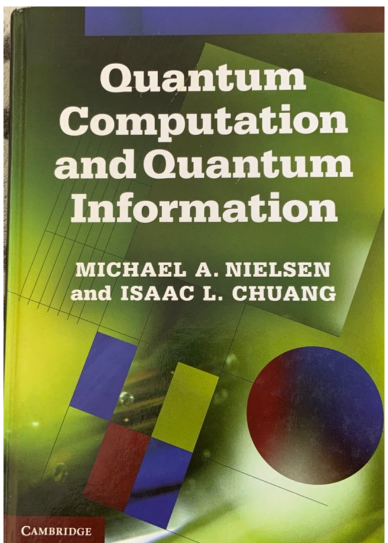
Meeting 5.1: Quantum Mechanics

I. The postulates

Section 2.2 of Nielsen + Chuang



I. Postulates of quantum mechanics

1. What are quantum states?
2. How can quantum states evolve over time?
3. How do we measure quantum states and how are quantum states affected by measurement?
4. What do composite systems look like?

The axioms specify the mathematical framework for answering these questions. The work of a physicist is to understand, for a specific system, what the specific mathematical objects are.

We will take a practical, mathematical approach to the axioms, and ignore (at least for now) their physical justification (e.g. Bell inequalities)

Dirac bra-ket notation

Notation	Description
z^*	Complex conjugate of the complex number z. $(1+i)^* = 1 - i$
$\|\psi\rangle$	Vector. Also known as a *ket*.
$\langle\psi\|$	Vector dual to $\|\psi\rangle$. Also known as a *bra*.
$\langle\varphi\|\psi\rangle$	Inner product between the vectors $\|\varphi\rangle$ and $\|\psi\rangle$.
$\|\varphi\rangle \otimes \|\psi\rangle$	Tensor product of $\|\varphi\rangle$ and $\|\psi\rangle$.
$\|\varphi\rangle\|\psi\rangle$	Abbreviated notation for tensor product of $\|\varphi\rangle$ and $\|\psi\rangle$.
A^*	Complex conjugate of the A matrix.
A^T	Transpose of the A matrix.
$A^\dagger$	Hermitian conjugate or adjoint of the A matrix, $A^\dagger = (A^T)^*$. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$.
$\langle\varphi\|A\|\psi\rangle$	Inner product between $\|\varphi\rangle$ and $A\|\psi\rangle$. Equivalently, inner product between $A^\dagger\|\varphi\rangle$ and $\|\psi\rangle$.

Can use this notation for arbitrary linear maps

1. States are vectors in a Hilbert space

complete

Postulate 1: Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the *state space* of the system. The system is completely described by its *state vector*, which is a unit vector in the system's state space.

"pure states"

Often, but not always, a state space also has a preferred basis ("computational basis")

Warning: Unlike in classical computing
a state is not a list of coefficients! The
coefficients can be computed, but we will get to that...

Examples: qubits, qutrits, qupits, qudits...

A qubit is any quantum system w/ a 2-dimensional state space, typically with a preferred basis.

In other words, $\mathbb{C}^2 = \mathrm{span}_{\mathbb{C}} \underbrace{\{|0\rangle, |1\rangle\}}_{\text{orthonormal basis}}$ is a qubit.

(so $|0\rangle \neq 0$)

Qutrit: $\mathbb{C}^3 = \mathrm{span}\{|0\rangle, |1\rangle, |2\rangle\}$

Qupit: $\mathbb{C}^p = \mathrm{span}\{|0\rangle, \ldots, |p-1\rangle\}$, p prime

Qudit: $\mathbb{C}^d = \mathrm{span}\{|0\rangle, \ldots, |d-1\rangle\}$, any d.

Amplitudes

If $|0\rangle, \ldots, |d-1\rangle$ is an ONB and

$|\psi\rangle = \sum_{i=0}^{d-1} a_i |i\rangle$ is any nonzero vector, we

call the a_i's (unnormalized) quantum amplitudes.

Normalizing and projectivizing

If $|\psi\rangle$ is not unit length but is at least nonzero,

then $\dfrac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}}$ is a state.

We'll see shortly that scalar multiples can't be distinguished, so we could define state space as projective space.

Mixed states

These are classical mixtures of quantum states

2. Time evolution is a Unitary transformation

Postulate 2: The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2,

$$|\psi'\rangle = U|\psi\rangle . \tag{2.84}$$

"Global version"

"Infinitesimal version"

Postulate 2′: The time evolution of the state of a closed quantum system is described by the *Schrödinger equation*,

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle . \tag{2.86}$$

H is constant in time for us

In this equation, $\hbar$ is a physical constant known as *Planck's constant* whose value must be experimentally determined. The exact value is not important to us. In practice, it is common to absorb the factor $\hbar$ into H, effectively setting $\hbar = 1$. H is a fixed Hermitian operator known as the *Hamiltonian* of the closed system.

Examples of unitary time evolution

For a qubit

$$\underbrace{I = \begin{array}{c|cc} & |0\rangle & |1\rangle \\ \hline |0\rangle & 1 & 0 \\ |1\rangle & 0 & 1 \end{array} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\text{Pauli operators}}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

H for "Hadamard," not Hamiltonian!

X also called "bit flip"

$$X|1\rangle = |0\rangle, \quad X|0\rangle = |1\rangle$$

Z called "(relative) phase flip" (no classical analog)

$$Z|0\rangle = |0\rangle, \quad Z|1\rangle = -|1\rangle, \quad Z\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Hamiltonians and energy eigenstates (i.e., eigenvectors of Hermitian operators)

A Hamiltonian is simply a self-adjoint operator H.
($H^{\dagger} = H$), intended to encode the energies of states

Eigenvectors are called "energy eigenstates."

Recall: H is diagonalizable with real spectrum.

The eigenvalues of H are the "energy levels" of the system.

$$\text{Global version} \underset{\text{Integrate / Solve Schrödinger equation}}{\overset{\text{Differentiate}}{\rightleftarrows}} \text{Infinitesimal version}$$

If $\quad i\hbar \dfrac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$

$$\Longrightarrow \quad \frac{d|\psi(t)\rangle}{dt} = -\frac{iH}{\hbar}|\psi(t)\rangle$$

Unitary!

$$\Longrightarrow \quad |\psi(t)\rangle = \int_0^t -\frac{iH}{\hbar}|\psi(\tilde{t})\rangle \, d\tilde{t} = \boxed{e^{-\frac{itH}{\hbar}}}|\psi(0)\rangle$$

3. Measurements are certain collections of linear operators (***)

Postulate 3: Quantum measurements are described by a collection $\{M_m\}$ of *measurement operators*. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle \,, \tag{2.92}$$

and the state of the system after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \,. \tag{2.93}$$

The measurement operators satisfy the *completeness equation*,

$$\sum_m M_m^\dagger M_m = I \,. \tag{2.94}$$

Example: Measuring a qubit in computational basis

$$M_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(|0\rangle\langle 0|)^2 = |0\rangle\langle 0|0\rangle\langle 0| = |0\rangle\langle 0|$$

Note: $M_0^t = M_0 = M_0^2$ and $M_1^t = M_1 = M_1^2$

but this need not be the case for general measurements.

Completeness equation follows immediately

E.g. $|\psi\rangle = \frac{1}{\sqrt{5}}\left(2|0\rangle - 3|1\rangle\right)$

Prob $|\psi\rangle$ winds up in state 1:

$$\langle\psi|M_0^t M_0|\psi\rangle = \langle\psi|M_0|\psi\rangle$$
$$= \ldots$$

If $|\psi\rangle$ does wind up in state 1, then its state is

$$\frac{M_1|\psi\rangle}{\sqrt{\langle\psi|M_1^t M_1|\psi\rangle}} = |1\rangle.$$

Measurements can not detect global phase

M_m measured on $|\psi\rangle$ vs. $e^{i\theta}|\psi\rangle$:

$$\left(e^{i\theta}|\psi\rangle\right)^{\dagger} = \langle\psi| e^{-i\theta} \quad \text{so}$$

$$\langle\psi|\cancel{e^{-i\theta}} M_m^{\dagger} M_m \cancel{e^{i\theta}}|\psi\rangle = \langle\psi|M_m^{\dagger} M_m|\psi\rangle$$

Take-away: Since measurements can't distinguish scalar multiples of a state, we should consider them physically indistinguishable.

Measurements can reliably distinguish orthogonal vectors

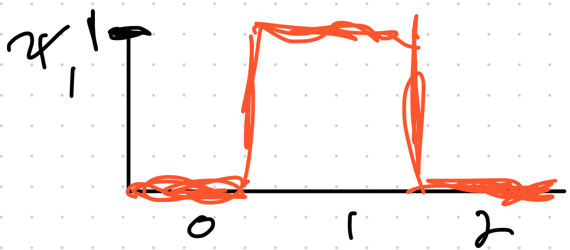
Given orthogonal normal states $|\psi_1\rangle$ and $|\psi_2\rangle$,

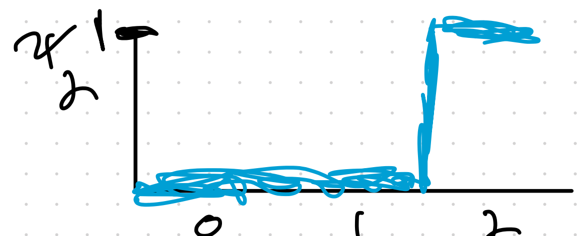
We can prepare measurements

$M_1 = |\psi_1\rangle\langle\psi_1|$, $M_2 = |\psi_2\rangle\langle\psi_2|$

and $M_0 = I - M_1 - M_2$.

You can check that the probability distributions are:





Measurements can probabilistically distinguish independent vectors

Box 2.3: Proof that non-orthogonal states can't be reliably distinguished

A proof by contradiction shows that no measurement distinguishing the non-orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ is possible. Suppose such a measurement is possible. If the state $|\psi_1\rangle$ ($|\psi_2\rangle$) is prepared then the probability of measuring j such that $f(j) = 1$ ($f(j) = 2$) must be 1. Defining $E_i \equiv \sum_{j:f(j)=i} M_j^\dagger M_j$, these observations may be written as:

$$\langle\psi_1|E_1|\psi_1\rangle = 1; \quad \langle\psi_2|E_2|\psi_2\rangle = 1. \tag{2.99}$$

Since $\sum_i E_i = I$ it follows that $\sum_i \langle\psi_1|E_i|\psi_1\rangle = 1$, and since $\langle\psi_1|E_1|\psi_1\rangle = 1$ we must have $\langle\psi_1|E_2|\psi_1\rangle = 0$, and thus $\sqrt{E_2}|\psi_1\rangle = 0$. Suppose we decompose $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\varphi\rangle$, where $|\varphi\rangle$ is orthonormal to $|\psi_1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, and $|\beta| < 1$ since $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal. Then $\sqrt{E_2}|\psi_2\rangle = \beta\sqrt{E_2}|\varphi\rangle$, which implies a contradiction with (2.99), as

$$\langle\psi_2|E_2|\psi_2\rangle = |\beta|^2\langle\varphi|E_2|\varphi\rangle \le |\beta|^2 < 1, \tag{2.100}$$

where the second last inequality follows from the observation that

$$\langle\varphi|E_2|\varphi\rangle \le \sum_i \langle\varphi|E_i|\varphi\rangle = \langle\varphi|\varphi\rangle = 1. \tag{2.101}$$

3! Measurements can be understood from "projective measurements" (***)

Projective measurements: A projective measurement is described by an *observable*, M, a Hermitian operator on the state space of the system being observed. The observable has a spectral decomposition,

$$M = \sum_m m P_m \,, \tag{2.102}$$

where P_m is the projector onto the eigenspace of M with eigenvalue m. The possible outcomes of the measurement correspond to the eigenvalues, m, of the observable. Upon measuring the state $|\psi\rangle$, the probability of getting result m is given by

$$p(m) = \langle\psi|P_m|\psi\rangle \,. \tag{2.103}$$

Given that outcome m occurred, the state of the quantum system immediately after the measurement is

$$\frac{P_m|\psi\rangle}{\sqrt{p(m)}} \,. \tag{2.104}$$

Statistics are easy to extract from projective measurements

Expectation value of M in state $|\psi\rangle$:

$$\mathbb{E}(M) = \sum_m m\, p(m) = \sum_m m \langle\psi|P_m|\psi\rangle$$

$$= \langle\psi|\sum_m m P_m|\psi\rangle$$

$$= \langle\psi|M|\psi\rangle.$$

Standard deviation:

$$\Delta(M)$$

$$= \mathbb{E}(M)^2 - \mathbb{E}(M^2) = \left(\langle\psi|M|\psi\rangle\right)^2 - \langle\psi|M^2|\psi\rangle$$

Heisenberg uncertainty: $\Delta(C)\Delta(D) \geq \dfrac{|\langle\psi|[C,D]|\psi\rangle|}{2}$

Proof: Use Cauchy-Schwarz....

Examples

Pauli operators! Every Hermitian operator on $\mathbb{C}^2$ is a $\mathbb{R}$-linear combination of Pauli operators.

$$aI + bX + cY + dZ$$

Hamiltonians are energy observables (that's why they control dynamics)

Allow us to answer questions like:
"Given state $|\psi\rangle$, what is the probability it has a certain energy?"

4. Composite systems are tensor products

Postulate 4: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

IF system A described by Hilbert space $\mathcal{H}_A$

" B " $\mathcal{H}_B$,

then the composite system AB is

$$\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B.$$

Curse of dimensionality (blessing?)

State space of n qubits is

$$\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = (\mathbb{C}^2)^{\otimes n}$$

dimension is 2^n.