Meeting 9.2: Toric code II, and Stabilizer Formalism

I. Toric code codevectors and distance

II. Stabilizer formalism

REMINDER

Construction

1. Put a qubit $\mathbb{C}^2$ on each edge $(2n^2)$

2. For each vertex v, define:

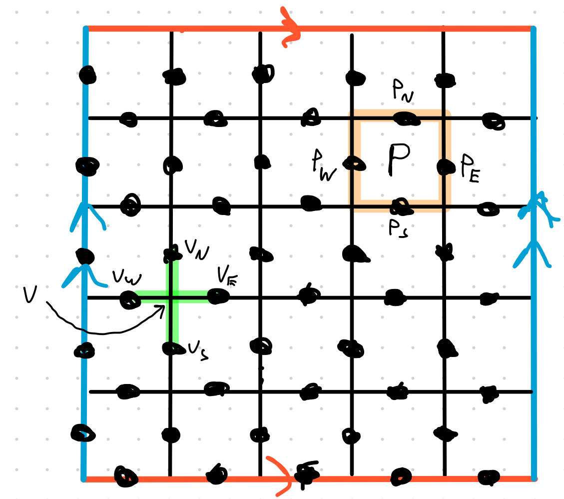
$$X_v = X_{v_N} X_{v_S} X_{v_E} X_{v_W}$$

(order doesn't matter)

For each 2-cell P ("plaquette"):

$$Z_P = Z_{P_N} Z_{P_S} Z_{P_E} Z_{P_W}$$

(order doesn't matter)



3. Codespace is

$$\mathcal{H} = \{ |\psi\rangle \in (\mathbb{C}^2)^{\otimes 2n^2} \mid X_v|\psi\rangle = Z_P|\psi\rangle = |\psi\rangle \quad \forall P, v \}$$

So, $\mathcal{H}$ is common +1 eigenspace of all vertex and plaquette operators

I. Toric code codevectors and distance

Finish claim from last time:

There's a basis of $\mathcal{H}$ "in natural" bijection with $H_1(S^1 \times S^1; \mathbb{Z}/2)$. The basis elements are equal superpositions of all cellular cycle representatives in given homology class.

Proof: Identify computational basis vector

$$|b_1 b_2 \cdots b_{2n^2}\rangle \in (\mathbb{C}^2)^{\otimes 2n^2}$$

with a $\mathbb{Z}/2$ cellular 1-chain. That is, $|b_1 b_2 \cdots b_{2n^2}\rangle$ encodes a formal $\mathbb{Z}/2$ linear combination of edges in cellulation of $S^1 \times S^1$.

From last time, given $|\psi\rangle \in \mathcal{H}$ with $\langle b_1 b_2 \cdots b_{2L^2} | \psi \rangle = c$, the condition $X_v |\psi\rangle = |\psi\rangle$ forces

$$\langle b_1 b_2 \cdots b_{2L^2} | X_{v_1} X_{v_2} \cdots X_{v_\ell} |\psi\rangle = c.$$

Moreover cycles $|b_1' b_2' \cdots b_{2L^2}'\rangle$ and $|b_1 b_2 \cdots b_{2L^2}\rangle$ satisfy

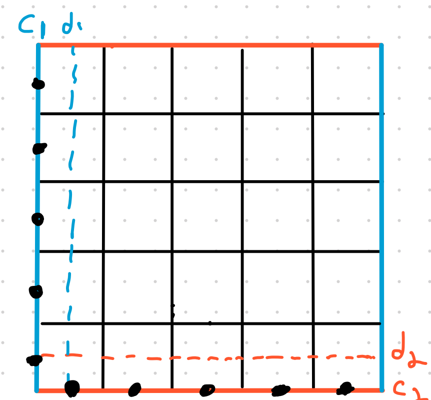
$$X_{v_1} X_{v_2} \cdots X_{v_\ell} |b_1 b_2 \cdots b_{2L^2}\rangle = |b_1' b_2' \cdots b_{2L^2}'\rangle$$

for some X's if and only if

$$\sum_{e \in c_1} b_e' = \sum_{e \in c_1} b_e \mod 2 \quad \text{and}$$

$$\sum_{e \in c_2} b_e' = \sum b_e \mod 2$$

where c_1 and c_2 as in here



$\square$

Distance?

Recall, a code can detect all k-qubit errors if and only if it can detect all products of X_i and Z_j supported on at most k qubits.

Suppose

$$E = X_{i_1}^{\alpha_1} X_{i_2}^{\alpha_2} \cdots X_{i_k}^{\alpha_k} Z_{i_1}^{\beta_1} Z_{i_2}^{\beta_2} \cdots Z_{i_k}^{\beta_k} \qquad (\alpha_j, \beta_j = 0, 1)$$

is such an error.

If E is a product of X_v's and Z_p's, then E acts trivially on $\mathcal{H}$, hence is not an error! If E takes $\mathcal{H}$ outside itself, then we can detect that because one of the X_v's or Z_p's will be violated.

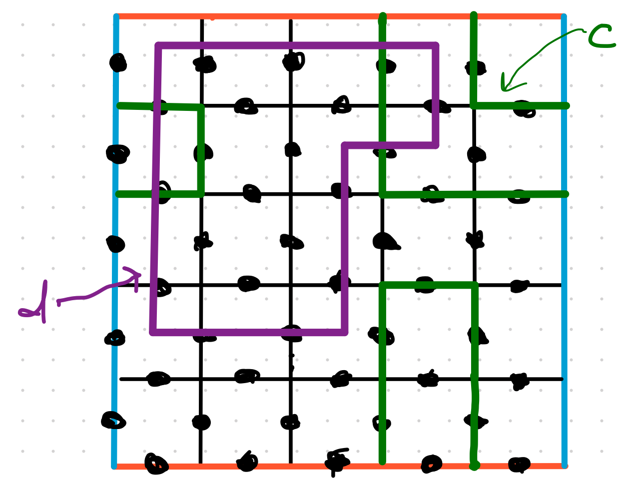
Main issue: if E preserves $\mathcal{H}$ setwise but not pointwise. That is, if $E|\mathcal{H}$ is nontrivial. In this case, we know from last time that $E|\mathcal{H}$ must be a product of loop or dual loop operators:

If c is a loop in 1-skeleton, define

$$Z_c = \prod_{e \in c} Z_e$$

If d a loop in dual 1-skeleton, define

$$X_d = \prod_{e \in d} X_e$$



Such a product can act nontrivially if and only if its support contains a cycle that is nontrivial in $H_1(S^1 \times S^1; \mathbb{Z}/2)$.

So,

$$\text{distance}(\mathcal{H}) = \min\{\#\text{supp}(c) \mid c \in Z_1^{cell}(S^1 \times S^1; \mathbb{Z}/2), [c] \neq 0 \in H_1\}$$

$$= n.$$

Toric code can be generalized to:

- k^{th} homology of any cell complex

- any $\mathbb{Z}/2$ chain complex.

II. Stabilizer formalism (after Calderbank - Rains - Shor - Sloane)

Toric code is an example of a stabilizer code.

Given n qubits, define error group

$$E = E_n \leq U(2^n) = U((\mathbb{C}^2)^{\otimes n})$$

to consist of all tensor products of form

$$\pm w_1 \otimes w_2 \otimes \cdots \otimes w_n \quad \text{or} \quad \pm i\, w_1 \otimes w_2 \otimes \cdots \otimes w_n$$

where $w_i = \mathrm{Id}, X, Y, \text{ or } Z$

Recall

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i X Z$$

E is a finite 2-group. 2^{2n+2}

Know if we can detect all errors from E supported on k qubits, then we can detect all errors from $U(2^n)$ supported on at most k qubits.

(More precisely, the stabilizer formalism presumes X, Y, Z each occur with same probability. However, without too much overhead, implies ability to correct errors in other models...)

Classical warm-up

Classical $\mathbb{Z}/2$ linear code is subspace $C \subseteq (\mathbb{Z}/2)^n$.

$(\mathbb{Z}/2)^n$ is of course space of all possible states,

but it is also space of all possible errors.

Error $e \in C$ iff e is undetectable (C is a subspace)

C corrects set of errors S iff for all

$s, t \in S$, either $s+t=0$

or $s+t \notin C$.

disjoint from S.

In quantum setting, a nontrivial operator may have trivial effect on codespace (e.g. X_v in toric code).

So, we look for <u>two</u> subgroups of E

$$S \leq S'$$

Analog of $\{0\}$, the "trivial errors"

Analog of C, the undetectable errors

For this to work, need S' to be centralizer of S. In particular, want S abelian. How can we construct?

(Compare toric code!)

$$\mathrm{Order}(E) = 2^{2n+2}$$

$$\mathrm{Center}(E) = C(E) = \{\pm I, \pm i I\}$$

$$\bar{E} := E/C(E) \cong (\mathbb{Z}/2)^{2n}$$

If $e \in E$, can uniquely write

$$e = i^{\lambda} X(a) Z(b)$$

where $\lambda \in \mathbb{Z}/4$ and

bitwise addition

$$X(a)|c\rangle = |a+c\rangle$$

(bit flip errors where $a_j \neq 0$)

$$Z(b)|c\rangle = (-1)^{b \cdot c}|c\rangle$$

(phase errors where

dot product mod 2

$a, b, c \in (\mathbb{Z}/2)^n$

If $e = i^{\lambda} X(a) Z(b)$, $e' = i^{\lambda'} X(a') Z(b')$, then

$$
\begin{aligned}
ee' &= i^{\lambda+\lambda'} X(a) Z(b) X(a') Z(b') \\
&= i^{\lambda+\lambda'} (-1)^{a' \cdot b} X(a) X(a') Z(b) Z(b') \\
&= i^{\lambda+\lambda'} (-1)^{a' \cdot b} X(a') X(a) Z(b') Z(b) \\
&= i^{\lambda+\lambda'} (-1)^{a' \cdot b} (-1)^{a \cdot b'} X(a') Z(b') X(a) Z(b) \\
&= (-1)^{a \cdot b' + a' \cdot b} \, ee'.
\end{aligned}
$$

So e and e' commute if and only if

$$a \cdot b' + a' \cdot b = 0. \ (\text{in } \mathbb{Z}/2) \qquad (*)$$

Write $\bar{e} = (a|b)$, $\bar{e'}$ $(a'|b')$ for images in $\bar{E}$.

e and e' commute iff $\bar{e}$ and $\bar{e'}$ orthogonal in $\bar{E}$ wrt $(*)$

$S \subseteq E$ will be abelian if and only if $\bar{e}, \bar{e}'$ orthogonal for all $\bar{e}, \bar{e}' \in \bar{S} \subseteq \bar{E}$.

In other words, S abelian if and only if $\bar{S}$ is totally isotropic.

Lagrangian

E.g. $\{ X(a) \mid a \in \mathbb{Z}/2 \}$ or $\{ Z(b) \mid b \in \mathbb{Z}/2 \}$.

Beware! $(*)$ is a symplectic inner product on $(\mathbb{Z}/2)^{2n}$

If $(a|b) \cdot (a'|b') = a \cdot b' + a' \cdot b$, then

$$(a|b) \cdot (a|b) = 2 a \cdot b = 0.$$

Centralizer of S is exactly the preimage of $\bar{S}^{\perp}$

(Note: $\bar{S} \leq \bar{S}^{\perp}$ if $\bar{S}$ is isotropic!)

Let codespace be

$$\mathcal{H} = \{ |\psi\rangle \mid e|\psi\rangle = |\psi\rangle \text{ for all } e \in S \}.$$

Define the <u>symplectic weight</u> of $(a|b) \in (\mathbb{Z}/2)^{2n}$ is # of nonzero pairs (a_i, b_i) when we write

$$(a|b) = (a_1, \ldots, a_n | b_1, \ldots, b_n)$$

Theorem If $\dim_{\mathbb{Z}/2} \bar{S} = n-k$ is isotropic wrt symplectic form, then $\mathcal{H}$ is a 2^k dimensional code with distance

$$d = \min_{v \in \bar{S}^\perp - \bar{S}} wt_{sym}(v).$$