Solovay-Kitaev Theorem (SU(2) version)

Thursday, 29 April, 2021 13:43

$\mathcal{G}$ is a finite gate set w/ inverses, $\langle \mathcal{G} \rangle$ dense in $SU(2)$

Thm: Any $U \in SU(2)$ can be approx'ed within dist ε using $O(\text{polylog}(1/\varepsilon))$ gates from $\mathcal{G}$.

$\log(1/\varepsilon)^c$ for some const c

Why care? Exists such $\mathcal{G}$ that is <u>fault-tolerant</u>

$\Rightarrow$ any quantum circuit w/ $f(n)$ gates can be approx'ed within dist ε w/ fault-tolerance w/ $O(f(n)\,\text{polylog}(f(n)/\varepsilon))$ gates.

Notation: $\mathcal{G}_\ell$ = length ℓ words from $\mathcal{G}$

$(\langle \mathcal{G} \rangle = \bigcup_{\ell \geq 0} \mathcal{G}_\ell)$

S ε-net of T means $\forall x \in T$ $\exists y \in S$ at most ε distance away

B_ε ball of radius ε around I.

distance tr norm $D(U,V) = \operatorname{tr}|U-V|$

$|X| = \sqrt{X^\dagger X}$

Step 1(a) approx I as well as desired.

depends on $SU(2)$ [Key Lemma: For sufficiently small ε, if $\mathcal{G}_\ell$ is ε^2-net for B_ε, then $\mathcal{G}_{5\ell}$ is $C\varepsilon^3$-net for $B_{\sqrt{C\varepsilon^3}}$

Since $\langle \mathcal{G} \rangle$ dense $\exists\, \ell_0$ & ε_0 s.t. $\mathcal{G}_{\ell_0}$ ε_0^2-net for B_{ε_0}.

By induction $\mathcal{G}_{5^k \ell_0}$ ε_k^2-net for B_{ε_k}

$$\varepsilon_k = \frac{(C\varepsilon_0)^{(3/2)^k}}{C} \qquad \text{wlog } \varepsilon_k^2 < \varepsilon_{k+1}.$$

Step 1(b) approx arbitrary U as well as desired.

$\langle \mathcal{G} \rangle$ dense so U_0 ε_0^2-approx of U

By induction: $U_k U_{k-1} \cdots U_0$ ε_k^2-approx of U

$U_i \in \mathcal{G}_{5^i \ell_0}$.

$U_i \in \mathcal{G}_{5^i l_0}$.

$$V = U(U_k U_{k-1} \cdots U_0)^{-1} = U U_0^\dagger U_1^\dagger \cdots U_k^\dagger$$

V is within dist ε_k^2 of I

$V \in B_{\varepsilon_{k+1}}$ $\quad \exists\, U_{k+1} \in \mathcal{G}_{5^{k+1} l_0}$ ε_{k+1}^2-approx of V.

$U_{k+1} U_k \cdots U_0$ ε_{k+1}^2-approx of U.

Total # of gates for ε_k^2-approx is $\sum_{i=0}^{k} 5^i l_0 = \frac{5^{k+1}}{4} l_0 = O(5^k)$

For desired approx ε, find $\varepsilon_k^2 < \varepsilon$

$$\frac{(C\varepsilon_0)^{(3/2)^k}}{C} < \varepsilon$$

const

$$\left(\frac{3}{2}\right)^k \cancel{\frac{2}{c}\log\left(\frac{1}{c\varepsilon_0}\right)} < \log\left(\frac{1}{\varepsilon}\right)$$

$$\left(\frac{3}{2}\right)^k = O\left(\log\left(\frac{1}{\varepsilon}\right)\right)$$

$$\log_{3/2} 5 \approx 4$$

$$\Rightarrow 5^k = O\left(\log\left(\frac{1}{\varepsilon}\right)^4\right).$$

Step 0.

Key Lemma: For sufficiently small ε, if $\mathcal{G}_l$ is ε^2-net for B_ε, then $\mathcal{G}_{5l}$ is $C\varepsilon^3$-net for $B_{\sqrt{C\varepsilon^3}}$

Step 0(a) If $\mathcal{G}_l$ is ε^2-net for B_ε then $\mathcal{G}_{4l}$ is ε^2-net for B_{ε^2}

Use Fact about $SU(2)$:

(1) if $U \in SU(2)$ $\quad U = u(\vec{a}) := \exp(-i\,\boxed{\vec{a}\cdot\vec{\sigma}}/2)$

real vector

vect. of Pauli matrices

(2) $[\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}]$

$[A,B] = AB - BA$

(2) $[a\cdot\sigma, b\cdot\sigma]$ $[A,B] = AB-BA$

$= 2i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$

vector · Pauli matrices · $\vec{a}\cdot\vec{\sigma}$ is a matrix

If U is close to I then

(3i) $U = \exp(-iA)$ A hermitian w/ very small trace.

(3ii) $\exp(-iA)\exp(-iB)\exp(iA)\exp(iB) \approx \exp(-[A,B])$

group commutator

$O(\varepsilon^3)$ approx

taylor expand + small traces

Pick $U \in B_{\varepsilon^2}$. $U \overset{(1)}{=} u(\vec{a})$ $\|\vec{a}\| < \varepsilon^2$

Find Two vectors $\vec{b}$ & $\vec{c}$ s.t. $\vec{a} = \vec{b}\times\vec{c}$ $\|\vec{b}\|, \|\vec{c}\| < \varepsilon$

$\rightarrow u(\vec{b})\ u(\vec{c}) \in B_\varepsilon$

Find $\underbrace{u(\vec{x})\ u(\vec{y})}_{\in \mathcal{G}_\ell}$ ε^2-approx for $u(\vec{b}), u(\vec{c})$

$\rightarrow u(\vec{x}\times\vec{y})$ $O(\varepsilon^3)$-approx for $u(\vec{b}\times\vec{c}) = u(\vec{a})$.

$$U \overset{(1)}{=} u(\vec{a}) \approx u(\vec{x}\times\vec{y}) = \exp(-\underbrace{2i(\vec{x}\times\vec{y})\cdot\vec{\sigma}}_{[\vec{x}\cdot\vec{\sigma},\, \vec{y}\cdot\vec{\sigma}]}/4)$$

$$\overset{(2)}{=} \exp\left(-\left[\frac{\vec{x}\cdot\vec{\sigma}}{2}, \frac{\vec{y}\cdot\vec{\sigma}}{2}\right]\right)$$

$$\overset{(3ii)}{\approx} \exp(-i\vec{x}\cdot\vec{\sigma}/2)\ \ldots$$

$O(\varepsilon^3)$-approx

$$= u(\vec{x})\,u(\vec{y})\,u(\vec{x})^\dagger\,u(\vec{y})^\dagger$$

$$\in \mathcal{G}_{4\ell}$$

Step $O(\ell)$ $\rightarrow$ $\mathcal{G}_{5\ell}$ $O(\varepsilon^3)$-net for $B_{O(\varepsilon^{3/2})}$.

$U \in B_{O(\varepsilon^{3/2})} \subseteq B_\varepsilon$ $V \in \mathcal{G}_\ell$ ε^2-approx of U.

$UV^{-1} = UV^\dagger \in B_{\varepsilon^2}$ $\quad W \in \mathcal{G}_{4\ell}$ $\quad O(\varepsilon^3)$-approx for $UV^\dagger$

WV $\quad O(\varepsilon^3)$-approx for U

$\in \mathcal{G}_{4\ell} \cdot \mathcal{G}_\ell = \mathcal{G}_{5\ell}.$

$\square$