1 Introduction

We are going to study the following differential equation:

\[ y(t) = y_0 + \sum_{k=1}^{\infty} \int_{t_k}^{t} V_i(y(s)) \, ds^i(s) \]  

(1.1)

Throughout the paper, we make the following assumptions:

1. The \( V_i \)'s are \( C^\infty \) vector fields on \( \mathbb{R}^d \) with bounded derivatives, and analytic on the set \( \{ y : \|y-y_0\| \leq C \} \) for some \( C > 0 \).

2. The driving path \( x : [0, T] \rightarrow \mathbb{R}^d \) is \( p \)-rough path with given approximating sequence \( \xi_n \) in \( C^{2\infty}(0, T; \mathbb{R}^d) \).

1.1 Motivation and development

when the driving process is fractional Brownian motion with Hurst parameter \( H \):


- H \( \in (1/2, 1/2) \) : smoothness of the density


- H \( \in (1/2, 1/2) \) : gradient bound B-Fouadou, C. Ouyang, S. Tindel-To appear.


Stochastic Taylor series


- Convergence and deterministic estimate for \( 1 \leq H < 3/4 \), P. Friz, X. Zhang-2012

- Asymptotic expansion (Castell estimate) for \( H < 3/4 \), P. Friz, X. Zhang-1993

2 Preliminary: Basic notations and facts of rough path theory

Let us introduce the following notations:

1. A word \( I = (i_1, \ldots, i_k) \in \{0, \ldots, d\}^k \) denote \( |I| = 1 \) is the size of \( I \), \( |I| \) number of 0 in \( I \).

2. For \( V_0, \cdots, V_{d+k} \) vector fields on \( \mathbb{R}^d \), \( V_I \) is the Lie derivative of the vector fields \( V_0, \cdots, V_{d+k} \)

\[ V_I = [V_0, V_I] + \cdots + [V_{d+k}, V_I] \]

3. Let \( x^I(t) \) be a \( n \)-dimensional rough path, and let us write for simplicity, \( x^I(t) = x^I \) for \( t \)

\[ \int_{0^I} \, ds^i = \int_{t_k^I}^{t_{k+1}^I} ds^i(t_k^I) \cdots ds^i(t_k^I) \]

4. It is invariant under permutation 

\[ \Lambda_{\sigma}(x) = \sum_{\sigma \in S_t^I} (-1)^{\sigma} x_{\sigma} \]

where \( \sigma \) is a permutation of \( i \) and \( \sigma(\cdot) \) is the set of all permutations of \( i \).

5. Taylor expansions and Castell estimates for solutions of stochastic differential equations driven by rough paths

6. We use the notation \( \| x^I \|_{\Delta, \{0, \ldots, d\}} \) as defined below

\[ \| x^I \|_{\Delta, \{0, \ldots, d\}} = \sup_{\delta(t)} \left( \sum_{i = 0}^{d+k} \left( \int_{t_k^I}^{t_{k+1}^I} ds^i(t_k^I) \right)^2 \right)^{1/2} \]

for some \( \Delta \), existence and smoothness of the density:

\[ \begin{aligned}
\int_{0^I} \, ds^i &= \int_{t_k^I}^{t_{k+1}^I} ds^i(t_k^I) \\
\| x^I \|_{\Delta, \{0, \ldots, d\}} &= \left( \int_{0^I} \, ds^i \right)^{1/2}
\end{aligned} \]

2.1 Taylor expansion for differential equations driven by \( p \)-rough paths

Definition 3.1. The Taylor expansion associated with the differential equation (1.1) is defined as

\[ y(t) = y_0 + \sum_{k=1}^{\infty} \int_{t_k^I}^{t_{k+1}^I} V_i(y(s)) \, ds^i(s) \]

(3.1)

where \( x^I(t) \) is the \( i \)-th projection path.

Next we prove the following general result:

Theorem 3.2. Let \( y_0 + \sum_{k=1}^{\infty} \int_0^t V_i(y(s)) \, ds^i(s) \) be the Taylor expansion associated with the equation (1.1) as defined in 3.1.

\[ y(t) = y_0 + \sum_{k=1}^{\infty} \int_{t_k^I}^{t_{k+1}^I} V_i(y(s)) \, ds^i(s) \]

is convergent and

\[ \| y(t) \|_{\Delta, \{0, \ldots, d\}} \]

Recall if \( I = (i_1, \ldots, i_k) \), we denote

\[ P^I = (V_1, \cdots, V_{d+k})^I \]

Theorem 3.3. Let \( \gamma \geq 1 \) and we assume that there exist \( M > 0 \) and \( C \) such that for every word \( I \in \{0, \ldots, d\}^k \)

\[ \| P^I \|_{\Delta, \{0, \ldots, d\}} \leq \gamma \|

(3.1)

For \( r > k \), we define \( T_{C^2}(r) = \inf \left\{ \sum_{k=1}^{\infty} \| I \|_{\Delta, \{0, \ldots, d\}} \right\} \)

(3.1)

2. There exists a constant \( Q_{C^2, \Delta, \{0, \ldots, d\}} \) depending on the subscript variables such that when \( t \leq T_{C^2}(r) \),

\[ y(t) = y_0 \sum_{k=1}^{\infty} \int_0^t V_i(y(s)) \, ds^i(s) \]

(3.1)

4 Castell expansion and tail estimate for differential equations driven by \( p \)-rough paths

Theorem 4.1. \( 0 \leq T_{C^2}(r) = T_{C^2}(r) \) and \( \gamma \geq 1 \) such that for \( t \leq T_{C^2}(r) \),

\[ y(t) = y_0 \sum_{k=1}^{\infty} \int_0^t V_i(y(s)) \, ds^i(s) \]

(3.1)

satisfying the following condition: these exist \( \gamma > 0 \), \( 0 < r \leq T_{C^2}(r) \)

(3.1)

The above theorem still holds with \( C_2 \) and \( \gamma \).

(4.2)

Remark 4.2. When the driving path \( x : [0, T] \rightarrow \mathbb{R}^d \) is a fractional Brownian motion with Hurst parameter \( H \), \( H < 1/2 \), \( x \) has a lift as a geometric \( p \)-rough path, \( p > 2 \), with given approximating sequence \( \xi_n \) in \( C^{2\infty}(0, T; \mathbb{R}^d) \).

References