In this talk (around 90 minutes), I will talk about stochastic analysis on manifolds. First, I will briefly give some background about diffusion process on manifold, including manifold-valued martingale and semi-martingale, horizontal lift, etc. Then, I will introduce the definition of Brownian motion on Riemannian manifolds by using the Laplace-Beltrami operator and talk about the relation between Laplace-Beltrami operator and Bochner's horizontal Laplacian. Second, I will present some classical result on the infinite dimensional path space on the Riemannian manifold, including quasi-invariance of Wiener measure, Clark-Ocone formula, integration by parts formula, Log-Sobolev inequality, etc. (If time available, I will also talk about similar result on sub-Riemannian manifold.)

1 Diffusion process on manifold

This draft is just a outline of the talk. And since I am trying to present most of the classical result in stochastic analysis on the path space of a Riemannian manifold, I will mainly state the result without any proof and talk about its idea and development. The materials in RED will be the main stream of the talk. The others will be presented depends on time and the audience.

1.1 Euclidean case

**Definition 1.1 (Brownian motion/theorem)** A Brownian motion $X$ in $\mathbb{R}^n$ can be defined as the diffusion process generated by the Laplace operator $\Delta = \frac{1}{2} \sum_{i=1}^{n} \left( \frac{\partial}{\partial x_i} \right)^2$. Which means it solve the martingale problem for $\Delta$, that is

$$M_t^f = f(X_t) - f(X_0) - \frac{1}{2} \int_0^t \Delta f(X_s) ds$$

is a martingale for smooth function $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

**Remark 1.2**

$$\Delta f = \lim_{t \to 0} \frac{\mathbb{E}(f(B_t)) - f(B_0)}{t}$$

In $\mathbb{R}^n$, the transition density function of Brownian motion is the Gaussian kernel:

$$p(t, x, y) = \left( \frac{1}{2\pi t} \right)^{n/2} e^{-|x-y|^2/2t},$$

we know that $p(t, x, y)$ solves the heat equation. $p(t, x, y)$ is the smallest positive solution of :

$$\frac{\partial p}{\partial t} = \frac{1}{2} \Delta u, \quad \lim_{t \to 0} p(t, x, y) = \delta_x(y)$$
The semigroup $P_t$ associated with the Brownian motion is defined as $P_t = e^{t\triangle}$ whose infinitesimal generator is $\triangle$ which satisfies

$$P_t f(x) = \mathbb{E} f(X_t) = \int_{\mathbb{R}^n} f(y)p(t,x,y)dy, \text{ for } x \in \mathbb{R}$$

Definition 1.3 (Martingale) In $\mathbb{R}^n$, a martingale is described as the fair game: $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$.

Definition 1.4 (Semimartingale) In $\mathbb{R}^n$, a semimartingale $Z_t = M_t + A_t$, where $M_t$ is a local martingale and $A_t$ cadlag adapted process with bounded variation.

Definition 1.5 (local semimartingale) A stochastic process $(M_t)_{t \geq 0}$ is called a local martingale (with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$) if there is a sequence of stopping times $(\tau_n)_{n \geq 0}$ such that:

1. The sequence $(\tau_n)_{n \geq 0}$ is increasing and almost surely satisfies $\lim_{n \to \infty} \tau_n = \infty$;
2. For $n \geq 1$, the process $(M_{t\wedge \tau_n})_{t \geq 0}$ is a uniformly integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

1.2 Manifold situation

Definition 1.6 (Path space on manifold) For a differentiable manifold $M$. An $M$-valued path $x$ with explosion time $\tau$ on $M$ is a subset of semimartingale which is invariant under local diffeomorphism. This means that smooth functions of semimartingales are semimartingale. The set of martingale is a subset of semimartingale which is invariant under local diffeomorphism.

Definition 1.7 (semimartingale on manifold) Let $M$ be a differentiable manifold and $(\Omega, \mathcal{F}, \mathbb{P})$ a filtered probability space. Let $\tau$ be an $\mathcal{F}_t$-stopping time. A continuous, $M$-valued process $X$ defined on $[0, \tau)$ is called an $M$-valued semimartingale if $f(X)$ is a real-valued semimartingale on $[0, \tau)$ for all $f \in C^\infty(M)$.

This means that smooth functions of semimartingales are semimartingale. The set of martingale is a subset of semimartingale which is invariant under local diffeomorphism.

Definition 1.8 (SDE on manifolds) An $M$-valued semimartingale $X$ defined up to a stopping time $\tau$ is a solution of SDE $(V_1, \ldots, V_l, Z, X_0)$ up to $\tau$ if for all $f \in C^\infty(M)$,

$$f(X_t) = f(X_0) + \int_0^t V_\alpha f(X_s) \circ dZ^\alpha_s$$

SDE $(V_1, \ldots, V_l, Z, X_0) : dX_t = V_\alpha f(X_s) \circ dZ^\alpha_s$

Definition 1.9 (diffusion process) (i) An $\mathcal{F}_t$-adapted stochastic process $X : \Omega \to W(M)$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a diffusion process generated by $L$ if $X$ is $M$-valued $\mathcal{F}_t$ semimartingale up to $e(X)$ and

$$M^f(X_t) = f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds, \quad 0 \leq t \leq e(X)$$
is a local $F_s$--martingale for all $f \in C^\infty(M)$.  

(ii) A probability measure $\nu$ on the standard filtered path space $(W(M), B(W(M))_s)$ is called a diffusion measure generated by $L$ if 

$$M^f(\omega)_t = f(\omega_t) - f(\omega_0) - \int_0^t Lf(\omega_s)ds, \ 0 \leq t \leq e(\omega),$$

is a local $B(W(M))_s$-martingale for all $f \in C^\infty(M)$.  

The elliptic operator and hypoelliptic operator admits invariant measure. $\mu P_t f = \mu f, P_t = e^{tL}$.

**Definition 1.10 (Laplace-Beltrami operator)** The Laplace-Beltrami operator $\triangle_M$ on a Riemannian manifold is:  

$$\triangle_M f = \text{div}(\text{grad } f)$$

In local coordinates representation:

$$\triangle_M f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} g^{ij} \frac{\partial f}{\partial x^j}) = g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + b^i \frac{\partial f}{\partial x^i} \ \ b^i = \frac{1}{\sqrt{G}} \frac{\partial (\sqrt{G} g^{ij})}{\partial x^i}$$

A general SDE in Stratonovich form: $dX_t = V_\alpha(X)_t \circ dW_t^\alpha + V_0(X_t)dt$  

It generates a diffusion process with infinitesimal generator:  

$L = \frac{1}{2} \sum_{i=1}^l V_\alpha^2 + V_0$

### 1.3 Brownian motion on Riemannian manifold

Suppose $\mathbb{M}$ is a Riemannian manifold with Levi-Civita connection $\nabla$ and Laplace-Beltrami operator $\triangle_M$. Given a probability measure $\mu, \exists! \triangle_M/2$-diffusion measure $\mathbb{P}_\mu$ on $(W(M), B_s)$. Any $\triangle_M/2$-diffusion measure on $P(\mathbb{M})$ is called a Wiener measure on $P(\mathbb{M})$.

Roughly speaking: Brownian motion on $M$ is any $M$-valued stochastic process $X$ whose law is a Wiener measure on the path space $W(M)$. Now let’s see how to construct the Brownian motion on a Riemannian manifold.

**Definition 1.11 (frame bundle)** A frame at $x \in \mathbb{M}$ is $\mathbb{R}$--linear isomorphism $u: \mathbb{R}^d \rightarrow T_x \mathbb{M}$ by sending $e_i \mapsto ue_i$. The frame bundle $\mathcal{F}(M) = \bigcup_{x \in \mathbb{M}} F(M)_x$, where $F(M)_x$ denotes the space of all frames at $x$.

There is a canonical projection $\pi: \mathcal{F}(M) \rightarrow M$ with $\pi u = x$. A tangent vector $X \in T_u F(M)$ is called vertical if it is tangent to the fiber $F(M)_{\pi u}$. We denote $V_u F(M)$ as the space of vertical vectors.  

A curve $\{u_t\}$ is horizontal if each $e \in \mathbb{R}^d$, the the vector field $\{u_t e\}$ is parallel along $\{\pi u_t\}$. A tangent vector $X \in T_u F(M)$ is horizontal if it is the tangent vector of a horizontal curve $u$.

We have the relation:  

$$T_u F(M) = V_u F(M) \otimes H_u F(M)$$

If we consider the orthonormal frames of the tangent space:

$$T_u \mathcal{O}(M) = H_u \mathcal{O}(M) \otimes V_u \mathcal{O}(M)$$

The projection $\pi: \mathcal{O}(M) \rightarrow M$ induces an isomorphism $\pi_*: H_u \mathcal{O}(M) \rightarrow T_x M$. At each $u \in \mathcal{O}(M)$ $H_i$ is the unique horizontal vector in $H_u \mathcal{O}(M)$ whose projection is the $i$th unit vector $ue_i$ of the
orthonormal frame: $\pi^* H_i(u) = u e_i$, $H_i(u) \in H_u O(M)$ $H_i(u)$ is the horizontal lift of $u e_i$. (in general, $\forall e \in \mathbb{R}^d$)

$$dI_t = \sum_{i=1}^n H_i(I_t) \circ dB_i^t, I_0 = (x, u), \text{ with } \pi u = x$$

$\pi I_t = X_t$, the projection defines a Brownian motion on $M$, and this is the Itô map.

**Definition 1.12 (Bochner’s horizontal Laplacian)** $\Delta_{O(M)} = \sum_{i=1}^n H_i^2$ is called the Bochner’s horizontal Laplacian on $O(M)$:

$$\Delta_M f(X) = \Delta_{O(M)} (f \circ \pi)(u), \forall u \in O(M) \text{ with } \pi u = x.$$

1.4 quasi-invariance of wiener measure

We first explain this idea in $\mathbb{R}^n$, for $h \in P_0(\mathbb{R}^n)$ is a absolutely continous and $\dot{h} \in L^2(I; \mathbb{R}^n)$. We define

$$|h|_H = \left( \int_0^1 |\dot{h}_s|^2 ds \right)^{1/2}$$

The finite energy space which we call the Cameron-Martin space is

$$\mathcal{H} = \{ h \in P_0(\mathbb{R}^n) : |h|_H < \infty \}$$

**Theorem 1.13 (Cameron-Martin-Maruyama)** Let $h \in \mathcal{H}$ and

$$\xi_t \omega = \omega + h, \omega \in P_0(\mathbb{R}^n)$$

a Cameron-Martin shift on the path space. Then the shifted Wiener measure $\nu^h = \nu \circ (\xi_h)^{-1}$ is absolutely continuous with respect to $\nu$ and

$$\frac{d\nu^h}{d\nu} = \exp[<h_s, \omega >_H - \frac{1}{2}|h^2|_H]$$

here

$$<h_s, \omega >_H = \int_0^1 <h_s, d\omega_s>$$

Cameron-Martin shifts are the only shift which preserves the measure class of the wiener measure. Then what is the analogue for this Quasi-invariance of Wiener measure on a (compact, complete, non-compact) Riemannian manifold. The key point is to prove a isometry representation theorem, which states that for a given $h \in \mathcal{H}$ starting from $x \in M$, what it will look like after the parallel transport associate with the given connection (e.g. Levi-Civitta connection, non-torsion free connection, etc.)

**Definition 1.14 (Directional derivatives)** The directional derivative of $F$ along $h$ should be defined by the formula

$$D_h F(\omega) = \lim_{t \to 0} \frac{F(\omega + t h) - f(\omega)}{t}$$

$$<DF, h> = D_h F$$
The above is the translating in the flat space, if we consider the translation on a Riemannian manifold, then the translation is associated with the connection, namely the parallel transport. One import thing is we want to estimate the gradient of the heat semigroup. We usually don’t have the commutation of the gradient and the semigroup.

**Theorem 1.15 (Gradient of the semigroup)** For $f \in C^\infty(M)$ we have

$$\nabla P_t f(x) = \mathbb{E}_x \{ M_T U_T^{-1} \nabla f(X_T) \}$$

where $M_t$ is the multiplicative functional:

$$\frac{dM_s}{ds} + \frac{1}{2} M_s \text{Ric} U_s = 0, \ M_0 = I$$

This is related to the gradient estimate and convergence to equilibrium for the kinetic Fokker-Planck semigroup $P_t = e^{tL}$, and other type parabolic equations. Does $P_t$ converge? In which sense? what is the speed?

$$\Gamma(f, g) = \frac{1}{2} (L(fg) - fLg - gLf)$$

and it’s iteration Bakry’s $\Gamma^2$

$$\Gamma^2(f, g) = \frac{1}{2} (L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(Lf, g))$$

The Bakry-Émery criterion:

$$\Gamma_2(f, f) \geq \rho \Gamma(f, f)$$

then the semi-group $P_t$ generated by the $L$(admits a symmetric measure) converge exponentially fast to equilibrium. $\int_M f d\mu / \mu(M)$. That

$$\text{Ent}_\mu(P_t f) \leq e^{-2\rho t} \text{Ent}_\mu(f)$$

**Remark 1.16**

- Ricci-curvature lower bound–Bakry-Ledoux-Lott-Villani
- Metric-measure spaces

**Theorem 1.17 (Theorem 2.1 by Hsu)** Suppose that $\gamma \in W_0^\infty(M)$ and $h \in W_0^\infty(\mathbb{R}^d)$. The pullback $J^{-1}D_h$ of the vector field $D_h$ under the development map $J^{-1} : W_0^\infty(M) \to W_0^\infty(\mathbb{R}^d)$ is given at $\omega = J^{-1} \gamma$ by the following $\mathbb{R}^d$-valued function on $[0,1]$:

$$p_h(\omega)_s = h_s - \int_0^s \Theta U_\tau(Hd\omega_\tau, Hh_\tau) - \int_0^s K_h(\omega)_\tau d\omega_\tau,$$

where $U = U(\gamma)$ is the horizontal lift of $\gamma$ in $O(M)$ and

$$K_h(\omega)_s = \int_0^s \Omega U_\tau(Hd\omega_\tau, Hh_\tau).$$
\[ \xi_t^\omega = \omega + \int_0^t p_h(\xi^\omega_\tau) d\tau \]  

(1.2)

dthis is the unique flow generated by \( p_h \), then we have the quasi-invariance, the Radon-Nicodym derivative is

\[ \frac{d\mu^t_{p_h}}{d\mu} = \exp\left[ \int_0^1 C_s^\omega(\xi^{-t} B)^* dB_s - \frac{1}{2} \int_0^1 \left| C_s^\omega(\xi^{-t} B) \right|^2 ds \right] \]  

(1.3)

Remark 1.18 Moving frames for frame map \( u : \mathbb{R}^n \rightarrow T_x \mathbb{M} \), the \( \mathbb{R}^n \)-valued canonical one-form \( \theta(X) = u^{-1} \pi_s(\omega) \), then the first structure equation and second structure equation gives us the connection one form, torsion form and curvature form

\[ d\theta = -\omega \wedge \theta + \Theta \]
\[ d\omega = -\omega \wedge \omega + \Omega \]

Definition 1.19 (geometric flow equation) The geometric flow equation (associated to \( h \)) is the differential equation

\[ \dot{\sigma}(t) = H(\sigma(t)) h, \]  

(1.4)

where \( \sigma : \mathbb{R} \rightarrow W(M) \) is a path of semimartingales, and \( H(.) \) is the horizontal lift operator in Theorem 3.3 (Driver). We assume here that \( P-a.s. \) the function \((t, s) \rightarrow \sigma(t)(s) : \mathbb{R} \times [0, 1] \rightarrow M \) is \( C^{1,0} \).

Theorem 1.20 (Theorem 5.1 by Bruce Driver) Let \( \sigma : J \rightarrow W^\infty(M) \) be a \( C^1 \) function satisfying the above geometric flow equation, and set \( \omega(t) = I^{-1} \circ H(\sigma(t)) \). Then \( \omega : J \rightarrow W^\infty(M) \) is a \( C^1 \) function and satisfies

\[ \dot{\omega}(t) = \int C(\omega(t)) \delta \omega(t) + h, \]  

(1.5)

where for any Brownian semimartingale \( (\omega) \)

\[ C(\omega) \equiv A(\omega) + T(\omega) \]  

(1.6)

with

\[ A(\omega) \equiv \int \Omega_u \langle h, \delta \omega \rangle, \]  

(1.7)

and

\[ T(\omega) \equiv \Theta_u \langle h, \cdot \rangle, \]  

(1.8)

where \( u \equiv \pi \circ I(\omega) \).

1.5 Clark-Ocone formula

The Clark-Ocone formula:

Proposition 1.21 Given \( F = f(X_t, \cdots, X_{t_n}), f \in C^\infty_c(\mathbb{M}^n), x \in \mathbb{M} \) is the starting point of the process \( X_t \in W(\mathbb{M}) \), then

\[ F = \mathbb{E}_x(F) + \int_0^T \langle \mathbb{E}_x(\tilde{D}_s F | \mathcal{F}_s), //0, s dB_s \rangle_{\mathcal{H}}. \]

Remark 1.22 (Itô representation formal) 

\[ f(X_t) = P_t f(x) + \int_0^t < a_s, dB_s >_{\mathcal{H}} \]
1.6 integration by parts formula

Theorem 1.23 (IBP) Let $F, G$ be two cylindrical functional. Then

$$< D_h F, G > = < F, D^*_h G >$$

where

$$D^*_h = -D_h + I_h$$

Proof. we can write

$$\frac{d}{dt} F \circ \zeta^t_h |_{t=0} = D_h F$$

so

$$(D_h F, G) = \frac{d}{dt} (F \circ \zeta^t_h, G)$$

The derivative is evaluated at $t = 0$. We have $\zeta^{-t}_h \circ \zeta^t_h \gamma = \gamma$. The Law $\nu^t_h$ of $\zeta^t_h$ is equivalent ot $\nu$, by the change of variables $\zeta^t_h \gamma \mapsto \gamma$

$$(F \circ \zeta^t_h, G) = (F, G \circ \zeta^{-t}_h)_{L^2(\nu^t_h)} = (F, G \circ \zeta^{-t}_h \{ \frac{d\nu^t_h}{d\nu} \})$$

$$\frac{d}{dt} \{ \frac{d\nu^t_h}{d\nu} \} = I_h$$

1.7 Log-Sobolev inequality

Gross proved the following Log-Sobolev inequality

$$\mathbb{E}(F^2 \log F^2) - \mathbb{E}(F^2) \log \mathbb{E}(F^2) \leq 2 \mathbb{E}(|DF|_{\mathcal{H}}^2)$$

In the sub-Riemannian case,

Theorem 1.24 For every cylindric function $G \in \mathcal{C}$ we have the following log-Sobolev inequality.

$$\mathbb{E}_x(G^2 \ln G^2) - \mathbb{E}_x(G^2) \ln \mathbb{E}_x(G^2) \leq 2e^{3T(K+\frac{\kappa}{2})} \mathbb{E}_x \left( \int_0^T \| D^*_s G \|^2 ds \right).$$

Aaron Naber improved the above upper bound constant in the Riemannian case.