Lesson 12 - Applications of Quadratic Functions

- Word problems
  1) identify and name your variables (give units if applicable)
     ↳ this can involve <u>drawing a picture</u>
  2) identify pieces of information and <u>express them as equations</u>
  3) what is the goal / what is the question asking for?
     ↳ translate into a mathematical statement
  4) check that you have answered all parts of the question <u>in the correct format</u>

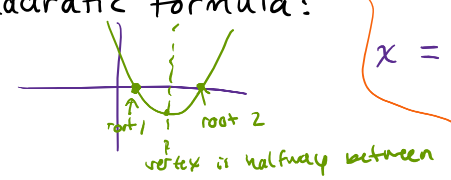
↳ depending on what the problem gives you, you may skip some steps (ex: they already gave you a picture or formula)

- Things to remember
  - max / min of quadratic function: vertex
    ↳ opens up / down
    $\cup$ $a>0$ $\cap$ $a<0$

    standard / vertex form
    $f(x) = a(x-h)^2 + k$
    (h, k)

    general form
    $f(x) = ax^2 + bx + c$
    $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

  - quadratic formula:

    $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

  - sketch graph of quadratic function using features
    ↳ opens up or down?
    ↳ axis of symmetry? x-coord. of the vertex

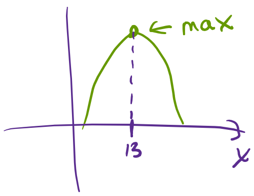
Example 1. The sum of two integers is 26. Find the values of these two integers so that their product is a maximum, and find the maximum value of the product.

1) identify and name your variables (give units if applicable) $\rightarrow$ two ints. : x and y
   ↳ this can involve drawing a picture
2) identify pieces of information and express them as equations $\rightarrow$ $x+y=26$
3) what is the goal / what is the question asking for? product: xy ← make this as large as possible
   ↳ translate into a mathematical statement

know: ax^2+bx+c

get rid of y: solve an eqn. for y & substitute

know: $x+y=26$ $\rightarrow$ $\boxed{y=26-x}$
$\quad -x \quad\quad -x$



product: $p(x) = xy = x(26-x) = 26x - x^2 = -x^2 + 26x + 0$

↑ maximize this

$a=-1 \quad b=26 \quad c=0$

vertex is at $x = -\frac{b}{2a} = \frac{-26}{2(-1)} = \frac{-26}{-2} = 13$

4) check that you have answered all parts of the question in the correct format

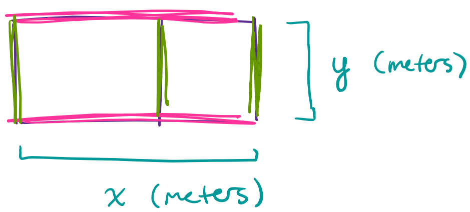
① x & y that maximize the product?
$x=13$ $\quad y = 26-13 = 13$

② max. value of xy?
$xy = (13)(13) = 169$

Example 2. A rectangular field is to be fenced off, and then divided in two by a fence running parallel to one of the sides. If 906 meters of fencing can be used, find the dimensions of the field that will maximize the total enclosed area, and then find the maximum area. Enter the smaller dimension first.

↳ draw a picture

area : length × width



Area : xy (meters^2)

$2x + 3y = 906$ m
$-2x \quad\quad -2x$

solve for y

$\frac{3y}{3} = \frac{906-2x}{3}$

$y = 302 - \frac{2}{3}x$

$A(x) = xy = x(302 - \frac{2}{3}x) = 302x - \frac{2}{3}x^2 = -\frac{2}{3}x^2 + 302x + 0$

TO DO: 1) x & y that maximize area?
2) maximum area?

$a = -\frac{2}{3}$, $b=302$: $x = -\frac{b}{2a} = \frac{-302}{2(-2/3)} = \frac{\cancel{302}^{151}}{2} \cdot \frac{3}{2} = \boxed{226.5 \text{ m}}$ ← x-coord of vertex → maximizes

1) $y = 302 - \frac{2}{3}(\frac{3}{2} \cdot 151) = 302 - 151 = \boxed{151 \text{ m}}$

2) $xy = (226.5)(151) = \boxed{34,201.5 \text{ m}^2}$

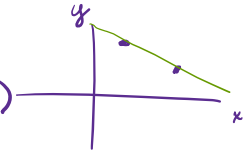
Example 3. A basketball arena can hold 9000 fans. When the ticket price is \$15, the average attendance is 7500. When the price drops to \$12, the average attendance rose to 7800. Assume that the attendance is linearly related to ticket price.

(a) What ticket price would maximize revenue?

(b) What is the maximium revenue? (price of ticket) x (# of tix bought)

$x \cdot f(x)$

(c) What is the attendance when the revenue is maximized?



$(15, 7500) = (x_0, y_0)$

$(12, 7800) = (x_1, y_1)$

x = price of ticket (\$)

y = f(x) = # of attendants

$f(x) = -100x + 9000$

Point-slope form: if (x_0, y_0) is on the line, then:

$$y - y_0 = \underbrace{m}_{\text{slope}}(x - x_0)$$

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{7800 - 7500}{12 - 15} = \frac{300}{-3}$$

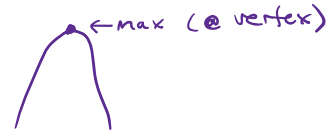
$m = -100$

$$y - 7500 = -100(x - 15)$$

$$y = -100(x-15) + 7500$$

$$= -100x + 1500 + 7500$$

$$= -100x + 9000$$



$$\text{revenue } r(x) = x \cdot f(x) = x(-100x + 9000) = -100x^2 + 9000x + 0$$

(a) x that maximizes r(x)

(b) maximum r(x)

(c) plug in result from 1) into f(x)

$a = -100$

$b = 9000$

$$x = -\frac{b}{2a} = -\frac{9000}{2(-100)} = \frac{90}{2} = \boxed{\$45} \quad \text{(a)}$$

(c) $f(45) = -100(45) + 9000 = 9000 - 4500 = \boxed{4500 \text{ people}}$

(b) $r(x) = x \cdot f(x)$

$$r(45) = 45(4500) = \boxed{\$202{,}500}$$

Example 4. The velocity of a subatomic particle moving through space can be modeled by

$$v(t) = \underset{a>0}{0.5}t^2 - 6.1t + 0.3$$

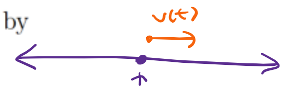
for $t \geq 0$ where t is time in seconds and v is velocity in m/s.

Find the following (Round all answers to 4 decimal places):

(a) The time(s) t at which the particle is not moving

(b) The interval(s) over which the particle is moving forward

(c) The interval(s) over which the particle is moving backward



$v(t) > 0$: particle is moving to right

$v(t) < 0$: particle is moving to left

$v(t) = 0$: particle is not moving

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$ax^2 + bx + c$

$$v(t) = 0 = 0.5t^2 - 6.1t + 0.3$$

$a = 0.5$

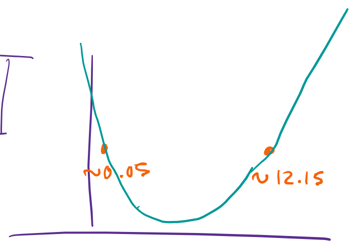
$b = -6.1$

$c = 0.3$

$$= \frac{6.1 \pm \sqrt{(6.1)^2 - 4(0.5)(0.3)}}{2(0.5)}$$

$$= \frac{6.1 \pm \sqrt{36.6}}{1}$$

(a) $\boxed{\begin{array}{l} t = 0.0494 \\ t = 12.1506 \end{array}}$



b) $\boxed{\begin{array}{l} t < 0.0494 \\ \text{or } t > 12.1506 \end{array}}$: $v(t) > 0$

c) $\boxed{0.0494 < t < 12.1506}$: $v(t) < 0$

Example 5. The height of a ball thrown from the top of a building can be modeled by the equation, where h is the height of the ball above the ground in feet and t is time in seconds.

$$h(t) = -16t^2 + 74t + 249$$

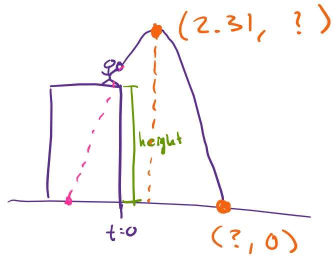
$a = -16 \quad b = 74$

↳ draw a picture

Find the following (Round all answers to 2 decimal places.):

(a) The height of the building.

$h(0) = -16(0)^2 + 74(0) + 249 = 249$ ft



(b) The time it takes for the ball to reach its maximum height.

vertex is at $t = -\frac{b}{2a} = \frac{-74}{2(-16)} = \frac{-74}{-32} = 2.31$ seconds

(c) The maximum height of the ball. $h(2.31)$ ft

$-16\left(\frac{74}{32}\right)^2 + 74\left(\frac{74}{32}\right) + 249 = 334.5625 \approx \boxed{334.56 \text{ ft}}$

(d) The time it takes for the ball to reach the ground.

choose +

$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = -16$
$b = 74$
$c = 249$

$h(t) = 0$

$\frac{-74 \pm \sqrt{(74)^2 - 4(-16)(249)}}{2(-16)} = \frac{-74 \pm 146.33}{-32}$

$\boxed{t = 6.89 \text{ sec}}$

$\cancel{t = -2.26}$ neg. ans. doesn't make sense