Lesson 13: Polynomials, Long Division, and Zeros

Factors of Polynomials

Ex 1: Let $P(x) = 2x^2 - 5x - 3$

ⓐ Evaluate $P(3)$ and $P(-\frac{1}{2})$

$$P(3) = 2(3)^2 - 5(3) - 3$$
$$= 18 - 15 - 3 = 0$$

$$P(-\tfrac{1}{2}) = 2(-\tfrac{1}{2})^2 - 5(-\tfrac{1}{2}) - 3$$
$$= \frac{2}{4} + \frac{5}{2} - 3$$
$$= \frac{1}{2} + \frac{5}{2} - 3$$
$$= \frac{6}{2} - 3$$
$$= 3 - 3 = 0$$

What is special about $P(3)=0$ and $P(-\frac{1}{2}) = 0$?

$\Rightarrow$ They are zeroes of $P(x)$.

$\Rightarrow$ $P(x)$ has the factors $(x-3)$ and $(x-(-\frac{1}{2}))$

ⓑ Factor $P(x) = 2x^2 - 5x - 3$

By part a, $P(x) = a(x-3)(x-(-\frac{1}{2}))$ → Checking my work

$$= 2(x-3)(x+\tfrac{1}{2})$$
$$= 2(x^2 - 3x + \tfrac{1}{2}x - \tfrac{3}{2})$$
$$= 2x^2 - 6x + x - 3$$
$$= 2x^2 - 5x + 3$$

True answer $P(x) = 2(x-3)(x+\frac{1}{2})$

If we can write a polynomial $f(x)$ as $f(x) = p(x)\,q(x)$, we say that $p(x)$ and $q(x)$ are factors of $f(x)$

The Factor Theorem

According to the Factor theorem, k is a zero of $f(x)$ $[f(k)=0]$ if and

The Factor Theorem

According to the Factor theorem, k is a zero of $f(x)$ $[f(k)=0]$ if and only if $(x-k)$ is a factor of $f(x)$.

This statement says 2 things

① If $f(x) = (x-k)p(x)$, then $f(k) = 0$

② If $f(k) = 0$, then $f(x) = (x-k)\,p(x)$

Polynomial Long Division

Ex 2: Divide the following polynomials using polynomial long division

(Note: there may be a remainder

ⓐ $(4x^2+3x-2) \div (2x+1) = \dfrac{4x^2+3x-2}{2x+1}$

$$
\begin{array}{r|l}
 & 2x + \tfrac{1}{2} \\
\hline
2x+1 & \cancel{4x^2} + 3x - 2 \\
 & -(\cancel{4x^2} + 2x) \quad \downarrow \\
\hline
 & \cancel{x} - 2 \\
 & -(\cancel{x} + \tfrac{1}{2}) \\
\hline
 & -\tfrac{5}{2}
\end{array}
$$

① Take the leading term inside and divide by the leading term outside

$$\frac{4x^2}{2x} = 2x$$

Put this over the term matching the order.

② Take what you got in ① and multiple it by the outside polynomial

$$2x(2x+1) = 4x^2 + 2x$$

Line your result up under the inside polynomial. Then subtract.

③ Repeat, and don't forget to bring the next term down

$$\frac{x}{2x} = \frac{1}{2}$$

$$\tfrac{1}{2}(2x+1) = x + \tfrac{1}{2}$$

$$\frac{4x^2+3x-2}{2x+1} = 2x + \frac{1}{2} + \frac{-5/2}{2x+1}$$

ⓑ $\dfrac{2x^3+3x^2-3x+18}{x+2}$

① $2x^3 = 2x^2$

(b) $\dfrac{2x^3+3x^2-3x+18}{x+3}$

$$
\begin{array}{r}
2x^2-3x+6 \\
x+3 \,\big)\,\overline{2x^3+3x^2-3x+18} \\
-(2x^3+6x^2) \\
\hline
-3x^2-3x \\
-(-3x^2-9x) \\
\hline
6x+18 \\
-(6x+18) \\
\hline
0
\end{array}
$$

① $\dfrac{2x^3}{x} = 2x^2$

$2x^2(x+3) = 2x^3+6x^2$

② $\dfrac{-3x^2}{x} = -3x$

$-3x(x+3) = -3x^2-9x$

③ $\dfrac{6x}{x} = 6$

$6(x+3) = 6x+18$

$$\frac{2x^3+3x^2-3x+18}{x+3} = 2x^2-3x+6$$

Side note: $(x+3)$ is a factor of $f(x)$.

$$2x^3+3x^2-3x+18 = (x+3)(2x^2-3x+6)$$

The Division Algorithm states that, given a polynomial dividend $f(x)$ where $\frac{f(x)}{d(x)}$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$ there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)\,q(x) + r(x)$$

where $q(x)$ is the quotient and $r(x)$ is remainder

Another way to divide polynomials is synthenic division.

Note: If the divisor is not of the form $(x-k)$, we must use long division.

Ex 3: Divide using synthetic division

(a) $\dfrac{2x^3+3x^2-3x+18}{x+3}$

$$
\begin{array}{r|rrrr}
 & x^3 & x^2 & x & c \\
-3 & 2 & 3 & -3 & 18 \\
 & & + & + & \\
 & & -6 & 9 & -18
\end{array}
$$

ⓐ $\dfrac{2x^3+3x^2-3x+18}{x+3}$

$x+3 = 0 \Rightarrow x = -3$

$= 2x^2-3x+6$

$$\begin{array}{r|rrrr} -3 & 2 & 3 & -3 & 18 \\ & \downarrow & + & + & \\ & & -6 & 9 & -18 \\ \hline & 2 & -3 & 6 & 0 \\ & x^2 & x & c & \text{remainder} \end{array}$$

ⓑ $\dfrac{x^3+2x^2+3}{x-1}$

Make sure you have all the terms. AKA if x^3, x^2, x, c

If not put me a zero in its place.

$= x^2+3x+3+\dfrac{6}{x-1}$

$$\begin{array}{r|rrrr} & x^3 & x^2 & x & c \\ 1 & 1 & 2 & 0 & 3 \\ & \downarrow & + & + & + \\ & & 1 & 3 & 3 \\ \hline & 1 & 3 & 3 & 6 \\ & x^2 & x & c & \text{remainder} \end{array}$$

Rational means fractions

The Rational Zero theorem

The Rational Zero Theorem states that if the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

has integer coefficients then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading term a_n

Ex 4: Let $f(x) = 6x^3-7x^2+1$.

ⓐ Use the rational zero theorem to list all possible rational zeroes for $f(x)$.

$$\frac{p}{q} = \frac{\text{factors of } 1}{\text{factors of } 6} = \frac{(\pm 1)}{(\pm 1, \pm 2, \pm 3, \pm 6)}$$

Possible combinations: $\cancel{\frac{1}{1}}\ 1,\ \cancel{\frac{1}{-1}}\ -1,\ \frac{1}{2}, \frac{1}{-2}, \frac{1}{3}, \frac{1}{-3}, \frac{1}{6}, \frac{1}{-6}$

Possible combinations: $\cancel{\frac{1}{1}}, \cancel{\frac{1}{-1}}, \frac{1}{2}, \frac{1}{-2}, \frac{1}{3}, \frac{1}{-3}, \frac{1}{6}, \frac{1}{-6}$

$\cancel{\frac{-1}{1}}, \cancel{\frac{-1}{-1}}, \cancel{\frac{-1}{2}}, \cancel{\frac{-1}{-2}}, \frac{-1}{3}, \frac{-1}{-3}, \frac{-1}{6}, \frac{-1}{6}$

$\Rightarrow 1, -1, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{3}, \frac{1}{3}, \frac{-1}{6}, \frac{1}{6}$

(b) Find all rational zeroes of $f(x) = \boxed{6}x^3 - 7x^2 + 1$

i.e. Plug in possibilities.

$f(1) = 6 - 7 + 1 = 0$ ✓ $\Rightarrow x = 1$ is a zero

$f(-1) = -6 - 7 + 1 = -12$ ✗

$f(\frac{1}{2}) = 0$ ✓ $\Rightarrow x = \frac{1}{2}$ is a zero

$f(-\frac{1}{3}) = 0$ ✓ $\Rightarrow x = \frac{-1}{3}$ is a zero

(c) Factor $f(x)$.

$$f(x) = 6(x-1)\left(x-\frac{1}{2}\right)\left(x+\frac{1}{3}\right)$$